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Symmetries and Interactions in Matrix String Theory

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Abstract

This PhD-thesis reviews matrix string theory and recent developments therein. Emphasis is put on symmetries, interactions and scattering processes in the matrix model. We start with an introduction to matrix string theory and a review of the orbifold model that flows out of matrix string theory in the strong YM coupling limit. Then we turn our attention to the appearance of U-duality symmetry in gauge models, after a (very) short summary of string duality, D-branes and M-theory. The last chapter reviews matrix string interactions and scattering processes in the high energy limit. Also, pair production of D-particles is studied in detail. D-pair production is expected to give important corrections to high energy scattering processes in string theory.

Symmetries and Interactions in Matrix String Theory

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Contents

Introduction

The subject of this thesis is a new effective formulation of string theory in terms of a $1+1$ dimensional Yang-Mills theory. This approach will prove to be useful for the investigation of the short-distance behavior of strings. Before we explain in some more detail what the coming chapters are about, we will first spend some words on the motivation to do string theory.

The motivation to study string theory is twofold: one is the desire of physicists to construct a theory that unifies all elementary forces in nature. A second important motivation is to learn more about gravity. Newtonian mechanics and general relativity give precise predictions for gravitational interactions at large distances. For example, we can in principle launch a spacecraft, explore a planet in our solar system and return safely to the earth again. Without our knowledge of gravity this would be a rather hazardous adventure.

For very tiny distance scales (like 10^{-33}cm, the Planck length) however, we expect that Newtonian mechanics or general relativity no longer gives an accurate description of gravity. The physical conditions at the Planck scale are so extreme that they cannot be realized in laboratories. This makes it hard to do actual experiments (but they are not ruled out). At present consistency is the only tool available . It is of interest to try to formulate a theory of gravity in the kinematic regime of the Planck scale, because it is relevant for the study of black holes (their existence is confirmed by convincing observational evidence) and the study of the early universe.

At the Planck scale quantum effects for gravitational interactions are important. When one tries to formulate a quantum field theory of gravity, for example by a perturbation expansion of the Einstein Hilbert action with a coupling term to a scalar field, increasingly severe short-distance divergencies appear, which cannot be removed by renormalization procedures. Therefore one has to find a way out; the singularities should be somehow smeared out by a drastic adaption of the theory.

At present there is only one consistent way known how to do this. This way is string theory, a theory where the fundamental objects are one dimensional, instead of zero dimensional as for point particles. Interactions in string theory have a geometrical interpretation in terms of smooth Riemann surfaces, as indicated in figure 1. This already gives a heuristic explanation why strings are able to smear out the short-distance divergences.

A key idea of string theory is that the different vibration modes of a string correspond to different particle states. The massless states are the most relevant ones, as they should form the particles known today in the standard model (which are massless or very light compared with the Planck mass). The effective theories of these states can in principle be derived from string theory, but in practice that can be hard.

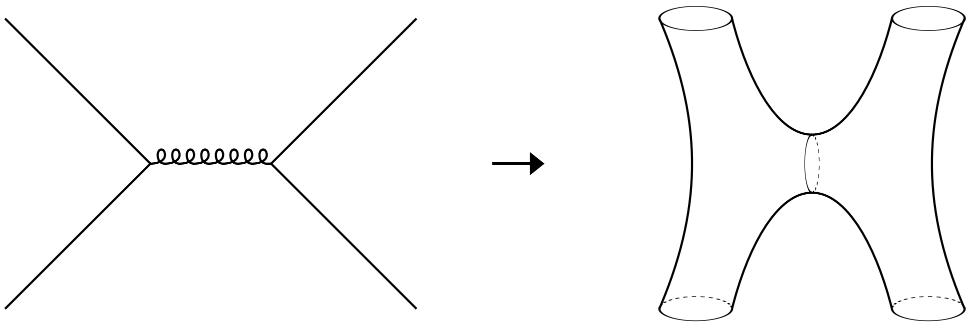

Figure 1 String theory smoothens out the short-distance divergences in field theories of gravity (like Einstein-Hilbert gravity coupled to a scalar field). The figure shows a tree level diagram in field theory which describes two scalar particles exchanging a graviton, and its analogue in string theory. Both diagrams are finite, but higher loop contributions are finite only in string theory.

It is not a priori clear why string theory should be the right way to proceed. One could for instance try membranes, objects with two spatial dimensions, which might be able to smear out the divergences as well. The study of membranes revealed some problems however, while string theory proved to have particular attractive features. Among these features are the following: string theory contains a massless spin-2 state (the graviton) whose low energy effective description is general relativity; string theory has a consistent perturbation expansion; it is rich enough to be a candidate for a unification of all known forces in nature. In particular string theory has intimate relations with gauge theories. There are no free parameters in string theory, instead there are many (classical) ground states parameterized by the expectation values the scalar fields take.

Consistency of string theory requires supersymmetry, a symmetry between the bosonic and fermionic degrees of freedom. Supersymmetry is expected to be a symmetry of nature for sufficient high energies only. The order of these energies is (almost) within reach of the elementary particle colliders of today, so in principle the superpartners of known particles states could be detected there. Detection of the superpartners would be an experimental proof of supersymmetry and strong evidence for string theory to be the right way to proceed.

Superstrings live in ten dimensions. This may be viewed as a drawback, but compactifications to lower dimensions give rise to a rich structure of Kaluza Klein fields. In ten dimensions we can define 5 superstring theories: type IIA and IIB, which are theories of closed strings; type I describing unoriented open strings (plus closed strings) and two types of the heterotic string. We will be mainly interested in type IIA and IIB so we will not dwell on the precise definition of other string theories. The important point here is that the theories are all related by duality transformations. It was conjectured by Witten that the five string theories are in fact manifestations of one eleven-dimensional theory, that goes with the name *M-theory*. The dualities and the existence of M-theory are not proven yet, but substantial evidence has been collected in recent years.

The string duality symmetries were known from earlier studies as *classical* symmetries of supergravity theories. These models failed to be consistent quantum theories of gravity; the addition of supersymmetry did not resolve the short-distance divergences. Later supergravity theories appeared again as low energy effective descriptions of the graviton state (and its superpartner the gravitino) of string theory and it is expected that their classical symmetries are genuine quantum mechanical duality symmetries in string theory.

An example of a string duality transformation is the mapping that relates strings at large distances to strings at small distances. This is somewhat similar to the more familiar electromagnetic duality in (supersymmetric) gauge theories, that relates the weak coupling to the strong coupling regime. Both duality transformations map a region of the theory where a perturbation expansion suffices, to a region where this expansion breaks down.

To understand (and to prove) these dualities we have to learn about the non-perturbative degrees of freedom. These degrees of freedom in string theory have been mysterious for a long time, till in the fall of 1995 Polchinski realized that the so-called D-branes carry them. D-branes were known before as hyper-surfaces in space-time on which open strings can end, but in fact they are dynamical non-perturbative objects in string theory.

At low energy the dynamics of D-branes are described by supersymmetric Yang-Mills theories. These theories have matrix valued scalar fields, that commute when the D-branes are widely separated. One can then diagonalize the matrices simultaneously and interpret the eigenvalues on the diagonals as the D-brane positions in the traditional sense. When the D-branes come close something remarkable happens: in general the matrices no longer commute and the interpretation of the D-brane positions gets obscured. Space-time becomes fuzzy: its coordinates no longer commute. This non-commutativity is somewhat similar to what happens with the classical phase space in quantum mechanics and is therefore highly suggestive.

The supersymmetric Yang-Mills model in $1+1$ dimensions compactified on a circle is an effective theory of one dimensional D-branes (called D-strings), but it can also be viewed as a theory of fundamental strings. This gauge theory, called matrix string theory, is defined on a cylinder that is covered by the world-sheet of a string one or more times.

Again one can think of the eigenvalues of the matrix valued scalars as the coordinates of the strings. These coordinates fields are not necessarily periodic along the Yang-Mills circle, but instead they may satisfy particular non-trivial boundary conditions. The simplest non-trivial example of a string of length 2 is illustrated in figure 2. In the lower left corner the two eigenvalues of a 2×2 matrix are mapped to each by going around the circle once. So though it looks like we have two short strings we in fact have one long string. In the lower right corner the configuration of eigenvalues has changed. Here the two eigenvalues of the matrix are periodic and we have two short strings. The one string configuration transforms to the other by changing the non-trivial boundary conditions at a certain time and place on the world-sheet. This is illustrated in the diagram in the middle of figure 2. We added in the figure the usual geometric representation of the interactions (the joining or splitting of strings).

Compared with the known perturbative string theories, matrix string theory has the advantage that non-perturbative degrees of freedom are contained in the model as well. Namely, the gauge model has besides matrix valued scalar fields, a two dimensional gauge field. This gauge field makes it possible to adorn the string states with an extra quantum number: the D-particle number (zero dimensional branes).

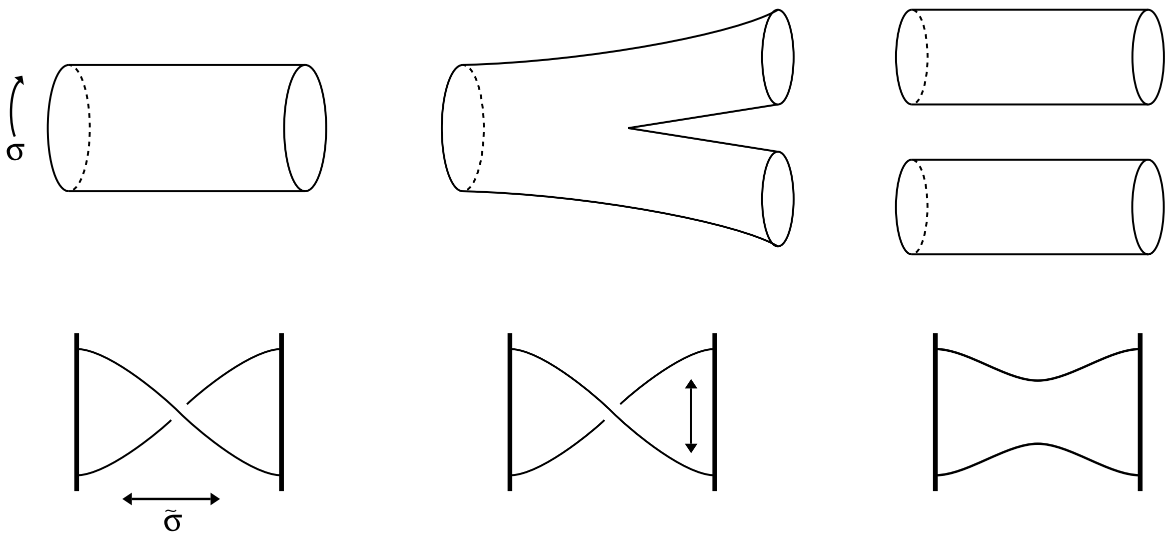

Figure 2 The smooth Riemann surfaces illustrate how a single closed string can split in two strings in perturbative string theory. The lower pictures show the eigenvalues of matrix valued scalar fields in matrix string theory as a function of $\tilde{\sigma}$, where $\tilde{\sigma}$ runs over *half* the interval of the spatial world-sheet coordinate σ. These eigenvalues describe the coordinates of strings. They may satisfy non-trivial boundary conditions on the $\tilde{\sigma}$ circle, so that the eigenvalues can describe different collections of strings. In the lower left corner we have one long string; in the lower right corner we have two short strings. These two configurations transform to each other when the boundary conditions of the eigenvalues are changed, as indicated by the arrow in the middlest diagram. The conventions are chosen such that the added string lengths are conserved during interactions.

D-particles are special, because they can be interpreted as Kaluza Klein particles in eleven dimensional M-theory compactified on a circle. It has been conjectured that the low energy effective model of D-particles (a $U(N)$ matrix quantum mechanics) is equivalent to M-theory in a particular gauge. This theory called matrix theory is closely related to matrix string theory.

The fact that matrix string theory contains the degrees of freedom of D-particles in a natural way, makes it possible to calculate non-perturbative corrections to processes in string theory. These corrections are important for the investigation of the short-distance behavior of strings, because in this regime the perturbation expansion of string theory appears to break down.

In chapter three we will show how string interactions can be realized in matrix string theory. Interactions arise as instanton type solutions of the Yang-Mills theory equations of motion. With these instantons known results from perturbative string theory can be reproduced. We will also start a calculation of non-perturbative corrections.

In chapter two we will investigate in how far the symmetries of string theory are present in (supersymmetric) gauge theories. In particular we will show that there are quantum states in the theory that have degeneracies consistent with a large string duality symmetry.

Chapter one reviews geometrical aspects of perturbative string theory in the light-cone gauge and in the discrete light-cone gauge quantization (DLCQ). We discuss a reformulation of DLCQ string theory in terms of an orbifold model. We end the chapter by introducing matrix string theory.

Although we have tried to add as much explanation of relevant basic concepts in string theory as possible, this thesis is not meant to be self-contained. As background material we refer to the textbooks on string theory [47][72] and quick introductions to light-cone string theory [37][38], to D-branes [71][7] and to matrix (string) theory [17][30]. For some basic concepts of conformal field theory we will need, we refer to [41].

Chapter 1

Towards Matrix String Theory

1.1 Perturbative string theory

1.1.1 The Polyakov approach to string theory

As strings move through space-time, they sweep out a two-dimensional surface. On this surface, called the world-sheet, the coordinates of the strings are defined. The history of a collection of strings is thus described by a map from the two-dimensional world-sheet into the d-dimensional space-time. This space-time is endowed with a metric that in principle should be derived from the string configuration. We will take the strings however in a fixed background metric, usually just flat.

In the Polyakov approach to string theory, the string perturbation expansion appears as a sum over two-dimensional Riemann surfaces. These Riemann surfaces are of a certain genus g (the counting parameter of the perturbation expansion) and have a number of boundary curves that correspond to the external states. They are the Feynman diagram representations of scattering amplitudes at g-loop order. An actual calculation of the contribution to the scattering amplitude in principle involves a path-integral over all maps of the surface into the space-time manifold, as well as an independent integration over all two-dimensional metrics g_{ab} on the genus g surface.

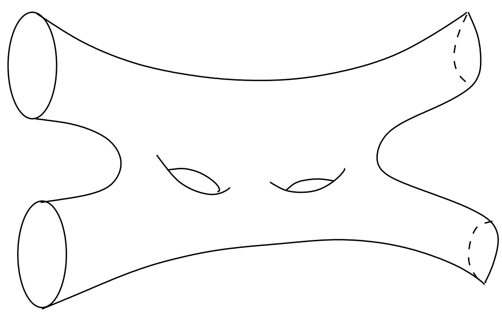

Figure 1.1 A typical world-sheet of genus two. The four tubes correspond with the external strings and extend to infinity.

More concretely in the Polyakov path integral approach, a general g-loop amplitude is given by the expression

$$A_g(\Psi_1, \cdots, \Psi_n) = \frac{1}{\mathcal{N}} \int \mathcal{D}g_{ab} \int_{\Psi_1, \cdots \Psi_n} \mathcal{D}X^\mu \exp(-S). \tag{1.1}$$

Here $\mathcal{N}$ is a normalization factor and the Ψ's indicate the external state wave functionals, that are defined on the boundaries of the world-sheet at infinity. For bosonic strings the action S is given by

$$S = \frac{1}{2\pi} \int d\sigma d\tau \sqrt{g} g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{1.2}$$

Here σ and τ are world-sheet coordinates taking values on a cylinder and $G_{\mu\nu}$ is the metric of space-time.

The action (1.2) has local symmetries, namely reparametrizations (diffeomorphisms) and Weyl rescalings. Weyl rescalings act on the world-sheet metric as

$$g_{ab} \to e^{\Omega} g_{ab} \tag{1.3}$$

and it easy to verify that the action (1.2) is invariant under this transformation.

One can use this Weyl invariance to make, at least locally, the world-sheet metric to be flat. Then there is still a gauge degree of freedom left over, namely arbitrary holomorphic coordinate transformations acting on the complex coordinate $w = \sigma + i\tau$ combined with an appropriate Weyl rescaling, so that the metric remains flat. These combined transformations are precisely conformal transformations, that will prove to be an important tool in string theory.

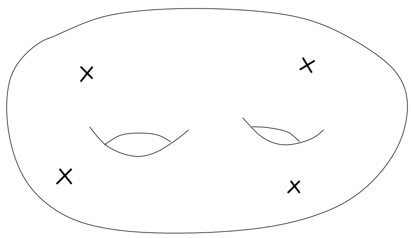

Figure 1.2 The world-sheet in figure 1.1 is conformally equivalent to a genus two Riemann surface with four punctures.

The most obvious parameterization of a world-sheet has boundaries at infinity (we assumed this up to now), like in figure 1.1. By a particular conformal transformation this surface can be mapped to a compact Riemann surface with punctures, that correspond with the external string states.

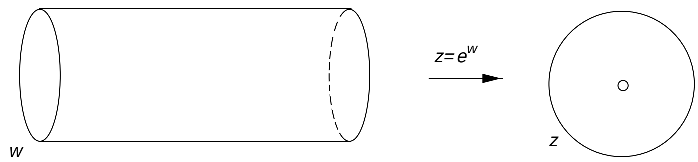

Figure 1.3 The conformal transformation $w \to z = e^w$, that maps cylinders to annuli in the complex plane.

This can be visualized by locally applying the exponential mapping that transforms a tube to an annulus in the complex plane. A world-sheet as in figure 1.1 is thus conformally equivalent to the Riemann surface in figure 1.2.

The conformal mapping that transforms the Riemann surface 1.1 to the one in figure 1.2 should be combined with the replacement of the boundary states $\Psi_i(X)$ by local operators V_i on the world-sheet. These operators called vertex operators create the external states on the world-sheet by introducing extra momentum and/or other quantum numbers. The most simple vertex function is the one that represents the emission of a tachyon, with no other degrees of freedom then the momentum. It is given by the expression

$$V = e^{ip\cdot X}, \tag{1.4}$$

where $p^2 = 8$ is the on shell condition for tachyons. In terms of the vertex operators the amplitude (1.1) becomes

$$A_g(1, \cdots, n) = \frac{1}{\mathcal{N}} \int \mathcal{D}g_{ab} \int \mathcal{D}X^\mu \exp(-S) V_1 \cdots V_n. \tag{1.5}$$

As mentioned before making use of the local symmetries of string theory greatly reduces the moduli space of world-sheet metrics over which one has to integrate. This simplification of the path integral is obtained after constructing a good slice through the space of metrics on the Riemann surface, that modulo Weyl transformations and diffeomorphisms covers all of moduli space only once. For external tachyons whose vertex operators are given by (1.4) the expectation value of the vertices is a simple Gaussian integral over the fields X, that can readily be calculated to be

$$A(1, \cdots, n) = g_s^{2g+2} \int [dm] \prod_i \int d^2 z_i \sqrt{g(z_i)} (\det{}'\mathrm{s}) \exp[-\frac{1}{2} \sum p_i \cdot p_j G_M(z_i, z_j)], \tag{1.6}$$

where $G_m(z_i, z_j)$ is the scalar Green function on the genus g world-sheet (depending on the moduli m, which have to be integrated over). In the integrand of (1.6) we abbreviated the fluctuation determinants that appear after integrating out the coordinate fields and reducing the path integral to a finite dimensional integral. In the limit that *all* $p_i \cdot p_j$ become large, the leading behavior of the integral can be derived by saddle point techniques [49]. Later on in chapter 3 these saddle points will prove to be crucial when we establish an alternative approach to string theory, called matrix string theory [65][10][29], that is also able to include non-perturbative corrections.

1.1.2 Light-cone string theory

In this section we will discuss some geometric facts of yet another approach to perturbative string theory: light-cone string theory. As background material we refer to [47][38] and [37].

Quantization of strings in the light-cone gauge formalism has two particular features. One is the absence of ghosts, only physical degrees of freedom are present, the other is the existence of a globally well-defined world-sheet time [39]. Covariance is broken in light-cone string theory, but the formalism has been proven to be equivalent to the covariant Polyakov path integral formulation of string theory [57].

In the light-cone gauge, light-cone time and world-sheet time are identified via

$$X^+(z,\bar{z}) = p^+\tau, \tag{1.7}$$

where p^+ is one of the light-cone momenta. This parameterization together with taking the world-sheet metric to be flat leads to the mass-shell condition

$$2p^+p^- = \int d\sigma((\partial_\tau X^I)^2 + (\partial_\sigma X^I)^2) := H, \tag{1.8}$$

with H the world-sheet Hamiltonian and p_- and p^+ the integrated light-cone momenta. The index I runs over all directions transversal to the light-cone directions.

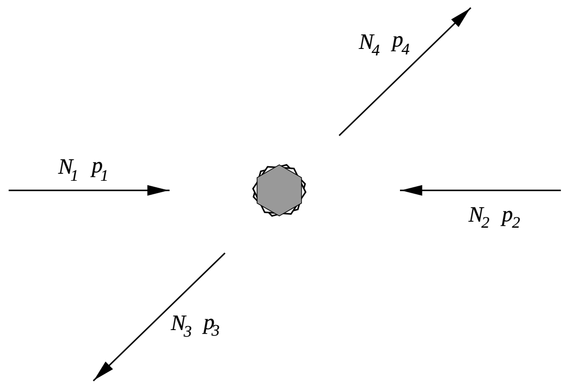

Figure 1.4 This figure indicates the kinematics of a two particle scattering process in string theory. The p_i are the eight dimensional transverse momenta that lie in one plane; the N's stand for the p^+ momenta.

A two particle scattering process like in figure 1.4 can be conveniently summarized in light-cone string theory by drawing a so called light-cone diagram or Mandelstam diagram. A light-cone diagram is a geometrical way of representing a scattering process in string theory. The external states are represented by tubes that extend to infinity. Together with a collection of other cylinders that correspond to internal strings they are glued together at interaction points. Thus a two-dimensional surface is formed that contains all essential information about the interaction process.

It is customary to rescale the spatial coordinate σ on the world-sheet, so that its range becomes $0 \leq \sigma \leq 2\pi p^+$. Because of this rescaling the radii α_i of the cylinders are proportional to the light-cone momenta p_i^+ of the corresponding strings. The total momentum p^+ is conserved during a scattering process, and therefore the sum of the radii of the cylinders is also conserved. An example of a light-cone diagram that describes the tree level contribution to the scattering process indicated in figure 1.4, is shown in figure 1.5 where we have two incoming strings that join to one string and split again into two strings. In the diagram the interaction times and possible twist angles are indicated. These twist angles are the angles under which the internal tubes are allowed to rotate before undergoing another interaction. As these twistings cannot be undone by a conformal transformation or reparametrization, they belong to the moduli of the light-cone string diagrams. [1]

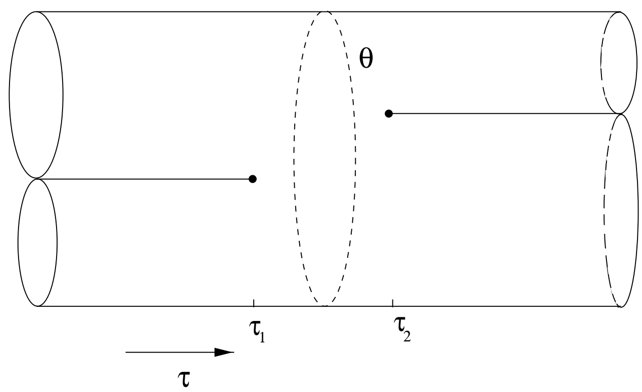

Figure 1.5 A tree level light-cone diagram. Two incoming strings join to one string and then split again into two outgoing strings. The interaction times τ are indicated, as well as one twist angle θ, over which the internal tube is allowed to rotate before undergoing another interaction.

For fixed external momenta p^+, a light-cone diagram is uniquely characterized by its moduli: the interaction times, the twist angles and the internal p^+ momentum fractions. In calculating the contribution to the scattering amplitude of a diagram like in figure 1.5 one has to do a path integral over these moduli. This path integral is equal to the analogous expression in the Polyakov approach (1.1). The general form of an h-loop contribution to the amplitude of n scattering strings is

$$A_n = \frac{1}{\mathcal{N}} \int [d\tau][d\alpha][d\theta] \int_{\Psi_1 \cdots \Psi_n} \mathcal{D}X^I e^{-S_{lc}}, \tag{1.9}$$

where the functional integral is taken over the moduli of the light-cone diagram. The integral is weighted with the exponential of the light-cone action that is just the action of free strings

$$S_{lc} = \int d\sigma d\tau (-(\partial_\tau X^I)^2 + (\partial_\sigma X^I)^2). \tag{1.10}$$

As discussed in the previous section light-cone diagrams like figure 1.5 are conformally equivalent to Riemann surfaces of genus equal to the number of internal strings, and a number of punctures on it, that correspond to the external strings.

An important feature of these Riemann surfaces is that the X^+ coordinate defines a one-form ω on it, that contains all relevant geometrical information. We can turn

[1] A twisting over an angle of 2π, called a Dehn twist, is a global symmetry of the light-cone string action. The twist angles are therefore defined modulo 2π.

this around: each Riemann surface has a unique one-form that can be used to define a local coordinate on the string world-sheet $w = \tau + i\sigma$ via

$$\omega = dw = dX^+(z). \tag{1.11}$$

This gives a precise relation between the parameterizations of the world-sheet z, w and the holomorphic component $X^+(z)$ of $X^+(z, \bar{z})$.

The abelian differential ω has simple poles at the punctures, with real residues that add up to zero. It moreover has purely imaginary periods on any homology cycle. Because the periods are purely imaginary the world-sheet time defined via (1.11) is well-defined, and can therefore be extended to all over the world-sheet of any genus [39]. The other coordinate σ in (1.11) is the multivalued space-like world-sheet coordinate.

The simple poles of the abelian differential (1.11) mark the locations of the vertex operators of the external states on the Riemann surface. This can be seen by noting that when the abelian differential (1.11) has a simple pole at a certain point w_i, with residue p_i^+

$$dw \sim \frac{p_i^+ dz}{(z - z_i)}, \tag{1.12}$$

we have for the local coordinate w in a neighborhood of the pole

$$w - w_i \sim p_i^+ \log(z - z_i). \tag{1.13}$$

Hence a punctured neighborhood of z_i in the complex plane is mapped by the logarithm to infinity, the inverse mapping of the conformal mapping illustrated in figure 1.3.

There are also specific points on the world-sheet at which strings join or split. These interactions take place at zeros of ω, that is critical points $z = z_0$ of the light-cone coordinate X^+. In the neighborhood of a simple zero of the abelian differential ω we have

$$dw \sim (z - z_0)dz \longrightarrow (w - w_0) \sim (z - z_0)^2. \tag{1.14}$$

From this equation we see that if we follow a contour around z_0 in the complex z-plane once, an angle 4π is swept out in w. This angle is the same one that is swept out when circling around an interaction point in the light-cone picture. We conclude that zeroes of ω correspond to points on the world-sheet where string interactions take place. Higher order zeroes correspond to higher order interactions.

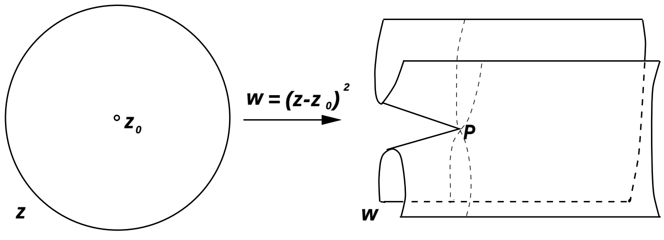

Figure 1.6 A critical point of ω corresponds with a string interaction point.

The abelian differential also contains information about the internal p^+ momenta and the twist angles via its integral along the a- and b-cycles of the Riemann surface. We have

$$\oint_{\alpha_i} \omega = i p_i^+, \qquad \oint_{\beta_i} \omega = i p_j^+ \mathcal{M}^i_{ja} \theta_a, \tag{1.15}$$

where $\mathcal{M}^i_{ja}$ is a real-valued matrix that can be chosen to have integer coefficients.

As an example for the location of the interaction points, we consider an n particle scattering process at tree level. For a given set of locations z^i of the corresponding vertex operators, the classical location of the world-sheet is described by

$$X^+(z,\bar{z}) = \frac{1}{2}\sum_i \epsilon_i p_i^+ \log|z - z_i|^2, \tag{1.16}$$

$$X^I(z,\bar{z}) = \frac{1}{2}\sum_i \epsilon_i p_i^I \log|z - z_i|^2, \tag{1.17}$$

where $\epsilon = 1$ for incoming and -1 for outgoing particles. The interactions take place at critical points $z = z_0$ of the light-cone coordinate X^+, cf (1.14)

$$dX^+ \mid_{z=z_0} = 0. \tag{1.18}$$

Inserting the explicit form (1.16) for X^+ gives

$$\sum_{i=1}^{n} \frac{\epsilon_i p_i^+}{z_0 - z_i} = 0 \ . \tag{1.19}$$

In case of n-point scattering, this condition can be reduced to an equation of degree $n-2$ in z_0, after a conformal transformation that maps one of the locations of the vertex operators to infinity. The $n-2$ zeroes correspond with elementary splittings or joinings of strings that occur at the interaction points of the corresponding tree level light-cone diagram.

1.1.3 High energy scattering in string theory

In chapter three we will be particularly interested in high energy four string scattering. The motivation to study string theory in this kinematic regime is to explore strings at short distances and to learn more about the structure of string theory. The high energy behavior of scattering strings has been studied in the past [3][49] by making use of perturbation techniques. An important result of this study, is that the dominant world-sheets at every order of the genus expansion, are all determined by the same saddle-point. As a consequence the dominant world-sheets have the same form at any genus (up to scaling factors), and their contribution to the scattering amplitude can be calculated in principle.

However the analysis revealed two apparent difficulties as well. One is that the size of scattering strings tends to grow with increasing energy. Strings are therefore not suitable as probes to investigate the short distance behavior of string theory. The other difficulty is the fact that the higher order amplitudes grow as an exponential of the energy. This behavior is completely different from most field theories. It leads to the disturbing conclusion that higher order corrections are important. This means, in less mild words, that the perturbation expansion breaks down for high-energy string scattering processes.

These two problems might be cured when we take non-perturbative effects into account. Firstly non-perturbative objects called D-branes can be used as probes for much smaller distance scales then the string scale α' [78][23][34]. Secondly non-perturbative effects will give important corrections to scattering amplitudes, as already was suspected in [49].

The need of a better knowledge of non-perturbative effects in string theory will be one of the main motivations to introduce the matrix string theory model. Before we do this, we will first explain some features of string theory in the so called discrete light-cone gauge (DLCQ) formalism. We will moreover review a reformulation of it in terms of an orbifold model.

1.1.4 String theory in the DLCQ formalism

In the DLCQ gauge one compactifies one of the light-like directions on a circle of radius R, so one identifies

$$X^- \sim X^- + 2\pi R. \tag{1.20}$$

The other light-like coordinate X^+ gets the interpretation of time. Compactification of X^- modifies the theory in two ways: the Hilbert space gets truncated to sectors with total p^+ momentum that is integer valued (in $1/R$ units)

$$p^+ = N. \tag{1.21}$$

In each sector the p^+ momenta of states take values in a finite positive set only (namely the integers $0 \ldots N$). Another modification of the theory is that we allow for winding modes around the X^- direction. These winding modes again decouple from the theory in the large N limit, which should lead us back to light-cone gauge string theory.

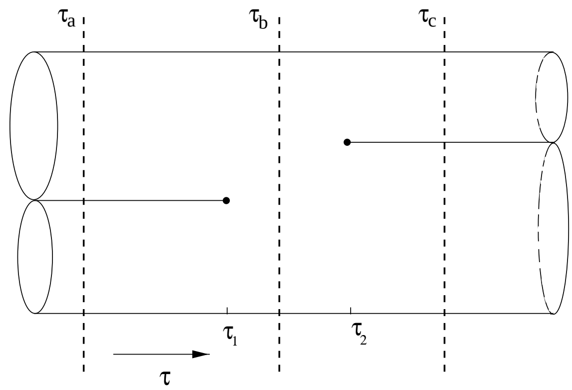

Figure 1.7 In this light-cone diagram we indicated three time slices. In the first time slice we have the two incoming strings, in the second the intermediate string and in the last slice we have the outgoing string configuration.

The fact that in DLCQ string theory p^+ momenta are integer valued, is an important first step towards defining matrix string theory. It will enable us to define matrices out of the string coordinates.

Consider the light-cone diagram, illustrated in figure 1.7 with total p^+ momentum equal to N. Out of the eight transversal coordinate fields of the strings we construct eight diagonal matrices in the following way.

The spatial world-sheet coordinate σ runs over an interval $[0, N]$. We cut this interval into N equal pieces of length one and define new fields $X_i^I(\sigma,\tau)$

$$X_i^I(\sigma,\tau) = X^I(\sigma + (i-1), \tau), \tag{1.22}$$

where i runs from $1, \cdots, N$ and $I = 1, \cdots, 8$ are the transversal directions and σ now runs over the interval $[0, 1]$. These fields are then interpreted as the eigenvalues of an $N \times N$ matrix

$$X^I(\sigma,\tau) = \begin{pmatrix} X_1^I(\sigma,\tau) & & & & \\ & X_2^I(\sigma,\tau) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & X_N^I(\sigma,\tau) \end{pmatrix}. \tag{1.23}$$

The light-cone action on the cylinder (1.10) becomes in terms of these matrix eigenvalues

$$S = \frac{1}{2\pi} \int d\sigma d\tau \left(\partial X_i^I \bar{\partial} X_i^I + i\theta_i^\alpha \partial \theta_i^\alpha + i\theta_i^{\dot{\alpha}} \bar{\partial} \theta_i^{\dot{\alpha}} \right), \tag{1.24}$$

where we included $16N$ fermionic fields $\theta_i^\alpha, \theta_i^{\dot{\alpha}}$ which together form N 16-component Majorana-Weyl spinors. The action (1.24) is left invariant by two space-time left-moving and two right-moving supersymmetries. The corresponding left-moving supercharge is

$$Q^\alpha = \sqrt{N} \oint d\sigma \sum_{i=1}^N \theta_i^\alpha, \qquad Q^{\dot{\alpha}} = \frac{1}{\sqrt{N}} \oint d\sigma G^{\dot{\alpha}}, \tag{1.25}$$

where

$$G^{\dot{\alpha}}(z) = \sum_{i=1}^N \gamma^I_{\alpha\dot{\alpha}} \theta_i^\alpha \partial X_i^I. \tag{1.26}$$

The right-moving charges have analogous definitions.

The eigenvalues in the matrix (1.23) are not single-valued, but multivalued with respect to the coordinate σ. For different time slices the eigenvalues X_i in (1.23) satisfy different quasi-periodic boundary conditions. For example we have for the first time slice τ_a two blocks of eigenvalues as indicated in the next equation

$$X^I(\sigma,\tau_a) = \begin{pmatrix} \boxed{\begin{matrix} X^I_1 & & \\ & \cdots & \\ & & X^I_{n_1} \end{matrix}} & \emptyset \\ \emptyset & \boxed{\begin{matrix} X^I_{n_1+1} & & & \\ & \cdots & & \\ & & \cdots & \\ & & & X^I_N \end{matrix}} \end{pmatrix}. \tag{1.27}$$

The collection of eigenvalues in the blocks each satisfy cyclic boundary conditions. The eigenvalues of the first block satisfy

$$\begin{aligned} X^I_i(\sigma+1) &= X^I_{i+1}(\sigma) \qquad i = 1\cdots n_1 - 1 \\ &, \\ X^I_{n_1}(\sigma+1) &= X^I_1(\sigma). \end{aligned} \tag{1.28}$$

We can write this condition as an $n_1 \times n_1$ matrix equation

$$X^I(\sigma+1) = VX^I(\sigma)V^{-1}, \tag{1.29}$$

with V the cyclic permutation matrix on the n_1 eigenvalues,

$$V = \begin{pmatrix} & 1 & & \emptyset \\ & & 1 & \\ & & & \ddots \\ & \emptyset & & 1 \\ 1 & & & \end{pmatrix}. \tag{1.30}$$

At time slice τ_c in figure 1.7 the blocks have changed; we then have an $n_3 \times n_3$ matrix and an $n_4 \times n_4$ matrix, representing the outgoing states with p^+ momenta n_3 respectively n_4.

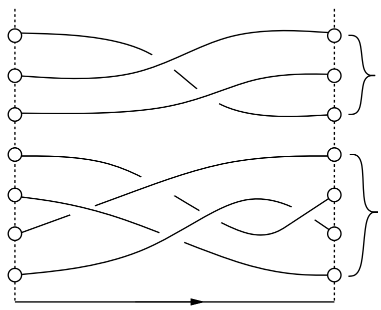

Figure 1.8 A configuration of long strings is determined by the particular boundary conditions on the fields X^I_i. Here we have two strings of length 3 and 4. Figure taken from [29].

For a general string configuration the coordinate matrix (1.23) will satisfy a periodicity condition of the form (1.29) with V a block diagonal matrix consisting of (say) s blocks of order n_i, such that each block can be taken of the form (1.30), and thus, as described above, defines a string of length n_i.

Each sector contributes to the total world-sheet energy and momentum via

$$H = \sum_i \frac{1}{n_i}(L_0^{(i)} + \bar{L}_0^{(i)}) \qquad P = \sum_i \frac{1}{n_i}(L_0^{(i)} - \bar{L}_0^{(i)}), \tag{1.31}$$

where we normalized the $L_0^{(i)}$ operators of each separate string, such that the oscillator levels are canonically counted. They moreover satisfy the level matching condition in the DLCQ formalism

$$L_0^{(i)} - \bar{L}_0^{(i)} = n_i m_i, \tag{1.32}$$

so that each contribution to the total world-sheet momentum is integer valued. Here m_i are the winding numbers along the light-like circle X^-. In the large N limit the usual level matching condition $L_0 - \bar{L}_0 = 0$ gets restored [29], because in this limit the winding states decouple. This arises, because when $N \to \infty$, only long strings survive with $p_+^i = n_i/N$ finite. These strings necessarily have zero winding number, because else they become infinitely massive compared to the energy of strings with vanishing winding number.

1.2 DLCQ STRING THEORY FROM AN ORBIFOLD MODEL

The numbers n_i that stand for the p^+ momenta of the separate strings in the preceding section, form a partition of total rank N. Thus they define a conjugacy class of the symmetric group S_N, that is the permutation group of N elements. One can write each group element g of S_N as a product of irreducible cyclic permutations. By conjugating g with an appropriate group element of S_N its (disjunct) cycle decomposition has the particular simple form

$$g = (1 \cdots n_1)(n_1{+}1 \cdots n_1{+}n_2) \cdots\cdots (N{-}n_k{+}1 \cdots N). \tag{1.33}$$

Other elements in the conjugacy class of g have an equivalent decomposition, that is their cycle decompositions all have the same number N_n of cycles of length n, with the obvious restriction

$$\sum_n n N_n = N, \tag{1.34}$$

so that the conjugacy classes of S_N are in one-to-one relation to partitions of N.

The now established fact that in the DLCQ formalism, one can associate to each collection of free strings, a conjugacy class of the symmetric group S_N, naturally suggests the following conjecture: free moving strings in the DLCQ sector $p^+ = N$ can be reformulated in terms of a supersymmetric two dimensional conformal field theory with $8N$ free bosons X_I^i, (here $i = 1 \ldots N$ and $I = 1 \cdots 8$) defined on the symmetric product orbifold

$$S^N \mathbb{R}^8 = (\mathbb{R}^8)^N / S_N, \tag{1.35}$$

that together with their fermionic partners are described by the action of free strings (1.24), where it is understood that all field configurations (X, θ) related by S_N transformations are identified $X \sim gX$, $\theta \sim g\theta$ for permutations g.

In the next subsections we will review evidence for this conjecture. First we will show that the model is able to recover the complete Fock space of second quantized type IIA string theory in the large N limit. Subsequently we will perturb the action of the model (1.35) by an appropriate interaction term that describes elementary splitting and joining of strings. After this we will review work that was done in [4] and [5], where the authors were able to reproduce all tree level four particle scattering amplitudes in string theory.

We will start however by making some general remarks about orbifold theories and their partition functions. The results we derive will be useful both for understanding matrix string theory, and for the next chapter, where we will calculate the degeneracy formula of BPS states in $\mathcal{N} = 4$ Yang-Mills theory on a three torus.

1.2.1 Hilbert space of an orbifold theory

Before we derive the Hilbert space of the orbifold model (1.35), we will first discuss the Hilbert space in the more general case of a conformal field theory defined on an orbifold target space M/G, that consists of a smooth manifold M divided out by a discrete group G. Again we take the two dimensional world-sheet of our model to be a cylinder parameterized by coordinates (σ, τ), where $0 < \sigma < 1$. As we divide out by the group G all field configurations (X, θ) are identified when they can be mapped to each other by a group element, $X \sim gX$, $\theta \sim g\theta$ for $g \in G$. Because of this identification the coordinate fields of the string are no longer necessarily periodic in the spatial coordinate σ. Instead it is allowed that they satisfy so called twisted boundary conditions

$$X(\sigma + 1) = gX(\sigma). \tag{1.36}$$

When we act with a group element h on the coordinate field X satisfying (1.36) we get another coordinate field $Y = hX$ with boundary condition $Y(\sigma+1) = hgh^{-1}Y(\sigma)$. The field Y is to be identified with X, so the twisted boundary conditions are well defined for conjugacy classes only. The group element g in (1.36) must therefore be thought of as a representative of its conjugacy class $[g]$.

In order to construct the G-invariant Hilbert space of the orbifold model, we first consider the subspace H_g which consists of states with sigma winding g (that is the coordinate fields satisfy the boundary condition (1.36)). The subspace H_g will be mapped to $H_{hgh^{-1}}$, when acting with a element h on the states, so we have to include the subsector $H_{hgh^{-1}}$ as well. In case that h is an element of the centralizer of g, H_g is mapped to itself. This implies that all states in H_g have to be invariant under the action of the centralizer C_g of g.

We thus conclude that the total Hilbert space of the orbifold model is determined by the conjugacy classes of the group G, with each twisted sector invariant under the centralizer. In other words the Hilbert space of an orbifold conformal field theory has the decomposition

$$\mathcal{H}(M/G) = \bigoplus_{[g]} \mathcal{H}_g^{C_g}(M), \tag{1.37}$$

where g is an arbitrary representative of the conjugacy class $[g]$ and where $\mathcal{H}_g^{C_g}$ is the g-twisted subsector of the total Hilbert space that is invariant under conjugation by elements in the centralizer C_g.

At the level of partition functions we can see the just described structure of the Hilbert space as follows. We start with the path-integral representation of the orbifold partition function on a torus with two homology cycles along spatial and time direction

$$Z = \frac{1}{|G|} \sum_{\substack{g,h\in G \\ [g,h]=1}} Z(g,h). \tag{1.38}$$

Here $Z(g,h)$ represents the partition function evaluated with twisted boundary conditions by the group elements g and h along the two different cycles. For consistency the elements g and h have to commute. The next step is then to recognize that the partition functions $Z(g,h)$ depend only on the conjugacy classes of G : $Z(xgx^{-1}, xhx^{-1}) = Z(g,h)$ for all $x \in G$. Hence we can write (1.38) as follows

$$Z = \sum_{[g]} \frac{1}{|C_g|} \sum_{h\in C_g} Z(g,h), \tag{1.39}$$

where we used the identity $|C_g||[g]| = |G|$. As the operator

$$P_{[g]} = \frac{1}{|C_g|} \sum_{h\in C_g} h \tag{1.40}$$

projects onto C_g invariant states we conclude that the partition function can indeed be written

$$Z = \sum_{[g]} \mathrm{tr}_{H_{[g]}} P_{[g]} q^{L_0} \bar{q}^{\bar{L}_0}. \tag{1.41}$$

The form of this partition function precisely corresponds with the structure of the Hilbert space of the orbifold model we have just discussed.

1.2.2 The long string picture

We have seen in the previous section that the Hilbert space of an orbifold model is determined by the conjugacy classes of the symmetry group that is divided out. For the symmetric product orbifold model (1.35) this group is the permutation group S_N.

As explained before a conjugacy class of the symmetric group S_N is labeled by a partition $\sum nN_n = N$ of N. The corresponding centralizer subgroup takes the form $C_g = \prod_n \mathbb{Z}_n^{N_n} \rtimes S_{N_n}$ where S_{N_n} permutes the N_n cyclic permutations of length n, and where each subfactor $\mathbb{Z}_n$ acts within one particular cyclic permutation. Due to the factorization of the conjugacy classes in irreducible elements of S_N, one can express the twisted sectors of the Hilbert spaces $\mathcal{H}_g^{C_g}$ as a product over the sectors of graded N_n-fold symmetric tensor products of smaller Hilbert spaces $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$

$$\mathcal{H}_g^{C_g} = \bigotimes_{n>0} S^{N_n}\mathcal{H}_{(n)}^{\mathbb{Z}_n}. \tag{1.42}$$

Combining the ingredients we get the decomposition for the total Hilbert space

$$\mathcal{H}(S^N X) = \bigoplus_{\substack{N_n \\ \sum_n nN_n = N}} \bigotimes_{n>0} S^{N_n}\mathcal{H}_{(n)}^{\mathbb{Z}_n}. \tag{1.43}$$

The space $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ is a particular $\mathbb{Z}_n$-invariant subsector of the total Hilbert space. It can be interpreted as the space of states of a single string living on $X \times S^1$, winded n times around the circle.

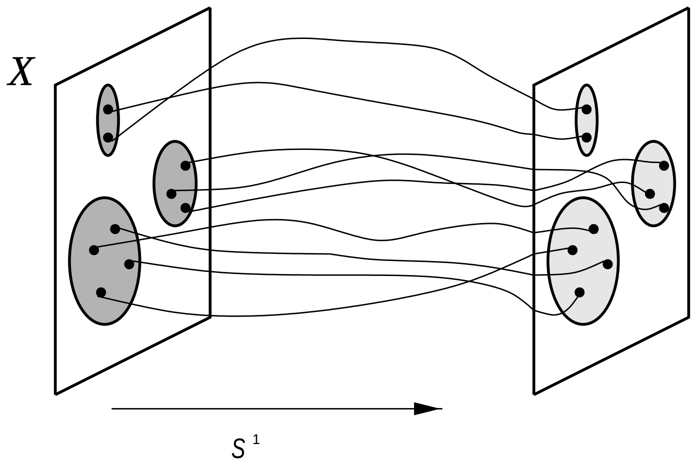

Figure 1.9 An illustration of the long string picture introduced in [28][29]. Three strings are shown, of length 2,3 and 4.

This single string has a description in n coordinate fields $X_I \in X$, with the cyclic boundary conditions (1.28) . Thus we are back to the string coordinate configurations of section 1.1.4. One can glue the coordinates X_I of a single string into one single field $X(\sigma)$ via

$$X(\sigma + k) = X_{k+1}(\sigma), \tag{1.44}$$

where now the argument of the field X runs over a circle of radius N.

This string coordinate has fractional oscillators modes, but invariance under the group $\mathbb{Z}_n$ implies that the fractional left moving minus right moving oscillator numbers add up to an integer. This can be easily seen by noting that $\mathbb{Z}_n$ acts by cyclic permutations on the fields X_I and therefore acts by translations $\sigma \to \sigma + 1$ on the field X of the single long string. In other words the $\mathbb{Z}_n$ invariance implies that the contribution from each single string sector to the total world-sheet momentum $P = L_0 - \bar{L}_0$ is integer-valued. So we are back to the decomposition of the total world-sheet energy and momentum in (1.31), combined with the level matching condition (1.32).

1.2.3 Twisted vacua and excitations

In this subsection we will discuss the twisted vacua of the model (1.35). It is convenient to change coordinates from the cylinder $w = \tau + i\sigma$ to the complex plane $w \to z = e^w$ (cf. figure 1.3).

A string of maximal length (i.e. a long string, whose length equals the total p^+ momentum) has coordinates that satisfy the quasi-periodic boundary conditions (1.29) . In the CFT we are considering these boundary conditions can be used to define a bosonic twist field $\sigma_n(z,\bar{z})$ [41] via the monodromy relation

$$X^I(e^{2\pi i}z, e^{-2\pi i}\bar{z})\sigma_n(0) = \omega_n X^I(z,\bar{z})\sigma_{(n)}(0), \tag{1.45}$$

where now X should be viewed as an operator and where ω_n is the generator of the $\mathbb{Z}_n$ group that acts cyclically on the long string sector of length n. The twist operator σ_n defined by (1.45) can be used to create the vacuum in the twisted sector by acting with it on the untwisted vacuum

$$|(n)\rangle = \sigma_{(n)}(0,0)\,|0\rangle\,. \tag{1.46}$$

In this twisted sector the left-moving component of the field $X(z,\bar{z})$ has the Laurent type power series expansion

$$\partial X^I_j = -\frac{i}{n}\sum_m \alpha^I_m e^{-\frac{2\pi i}{n}jm} z^{-\frac{m}{n}-1}, \tag{1.47}$$

where the α^I_m are the usual creation and annihilation operators that satisfy the commutation relation

$$[\alpha^I_m, \alpha^J_n] = m\delta^{IJ}\delta_{m+n,0}. \tag{1.48}$$

The conformal weight of the twist field $\sigma_{(n)}$ can be found by reading off the singular term in its OPE with the bosonic stress-energy tensor. This stress-energy tensor is defined as

$$T(z) = -\frac{1}{2}\sum_I^8 \sum_i^n :\partial X^I_i(z)\partial X^I_i(z):. \tag{1.49}$$

By a straightforward calculation one finds the conformal weight of $\sigma_{(n)}$. The result is

$$h^b_n = \frac{1}{3}(n - \frac{1}{n}). \tag{1.50}$$

The most singular term in the operator product expansion of the bosonic twist field $\sigma_{(n)}$ and the field ∂X^I_i is

$$\partial X^I_j(z)\cdot\sigma_{(n)}(w,\bar{w}) \sim (z-w)^{-(1-\frac{1}{n})} e^{\frac{2\pi i}{n}j}\tau^I_{(n)}(w,\bar{w}), \tag{1.51}$$

where

$$\tau^I_{(n)}(0,0)\,|0\rangle = -\frac{i}{n}\alpha^I_{-1}\,|(n)\rangle\,, \tag{1.52}$$

is the first excited state in the twisted sector $|(n)\rangle$. By dimensional counting we find that the conformal dimension of $\tau^I_{(n)}$ is $\frac{1}{3}(n+\frac{2}{n})$. The operator τ^I will be used later for the construction of the interaction vertex.

One obtains excitations of the twisted vacuum states by applying the usual vertex operators. For instance a scalar particle with momentum k^I in a sector $|n>$ is obtained by acting on this vacuum state with the vertex operator

$$: e^{ik_i^I X_i^I(0,0)} : |(n)>, \tag{1.53}$$

Here $k^I = \sum_{i=1}^n k_i^I$ is the total momentum of the collection of long strings in the transversal direction I.

In a long string sector of length n the expansions of the left and right-moving components of the fermions are

$$\theta_j^\alpha(z) = \frac{1}{\sqrt{n}} \sum_m \theta_m^\alpha e^{-\frac{2\pi i}{n}jm} z^{-\frac{m}{n}-\frac{1}{2}}. \tag{1.54}$$

$$\theta_j^{\dot\alpha}(\bar z) = \frac{1}{\sqrt{n}} \sum_m \theta_m^{\dot\alpha} e^{\frac{2\pi i}{n}jm} \bar z^{-\frac{m}{n}-\frac{1}{2}}. \tag{1.55}$$

The creation and annihilation operators satisfy the commutation relations

$$\left\{\theta_m^\alpha, \theta_n^\beta\right\} = \delta^{\alpha\beta}\delta_{m+n,0}. \tag{1.56}$$

These commutation relations imply that the zero modes form a Clifford algebra. This means that the vacuum state must represent this algebra. Because of triality in the transversal spatial index I and the spin indices α and $\dot\alpha$ the vacuum state can be chosen as a 16 component vector with components $|I\rangle$ and $|\dot\alpha\rangle$ normalized in the standard way, that moreover satisfy the relations [47]

$$\theta_0^\alpha\,|I\rangle = \frac{1}{\sqrt{2}}\gamma^I_{\alpha\dot\alpha}\,|\dot\alpha\rangle\,, \quad \theta_0^\alpha\,|\dot\alpha\rangle = \frac{1}{\sqrt{2}}\gamma^I_{\alpha\dot\alpha}\,|I\rangle\,. \tag{1.57}$$

The vacua $|I\rangle$ and $|\dot\alpha\rangle$ are created by primary fermionic twist fields (spin fields) which we denote by $\Sigma^I_{(n)}$ respectively $\Sigma^{\dot\alpha}_{(n)}$.

The most singular terms of the OPE of the fermionic twist fields and the fermionic fields θ^α are

$$\theta_i^\alpha(z)\cdot\Sigma^I_{(n)}(w) \sim \frac{1}{\sqrt{2n}}(z-w)^{-1/2}\gamma^I_{\alpha\dot\alpha}\Sigma^{\dot\alpha}_{(n)}(w), \tag{1.58}$$

$$\theta_i^\alpha(z)\cdot\Sigma^{\dot\alpha}_{(n)}(w) \sim \frac{1}{\sqrt{2n}}(z-w)^{-1/2}\gamma^I_{\alpha\dot\alpha}\Sigma^I_{(n)}(w). \tag{1.59}$$

The stress energy tensor for the fermionic fields is defined as

$$T^F(z) = -\frac{1}{2}\sum_{\alpha=1}^{8}\sum_{i=1}^{n} : \theta_i^\alpha(z)\partial\theta_i^\alpha(z) : . \tag{1.60}$$

From the OPE of the twist operator $\Sigma_{(n)}$ with the fermionic stress energy tensor one can read off its conformal weight.

It equals $\Delta_n^f = \frac{n}{6} + \frac{1}{3n}$.

Now we have defined the bosonic and fermionic twist operators we can construct vertex operators, that create the vacuum ground states of arbitrary twisted sectors. Each sector is represented by a particular conjugacy class of the symmetric group S_N. For a given conjugacy class $[g]$ the vertex operator can be written as a product of vertex operators in correspondence to the decomposition of a representative g in irreducible cyclic permutations of length n

$$V_{[g]} = \prod_n V_{(n)}. \tag{1.61}$$

Each vertex operator $V_{(n)}$ in equation (1.61) can be represented by products of bosonic and fermionic twist fields (1.51), (1.58)-(1.59), that together create the vacua of the twisted sectors. Explicitly the vertex operator (1.61) reads

$$V_{[g]} = \frac{1}{N!} \sum_{h \in S_N} V_{h^{-1}gh}(z, \bar{z}). \tag{1.62}$$

The expression (1.62) is invariant under conjugation of elements of the symmetric group and therefore well defined. We assume that the vertex operators can be written as the tensor product of a left-moving part and a right-moving part, that can each be decomposed in a fermionic twist operator and a bosonic operator.

Combining the fermionic and bosonic states, all the 256 massless states of IIA supergravity are obtained. For example in the long string sector a graviton with momentum k^I and polarization ζ is created by the vertex operator

$$V_{(n)}[k^I, \zeta](z, \bar{z}) = \zeta_{IJ} \sigma_{(n)}[k^I](z, \bar{z}) \Sigma^I_{(n)}(z) \bar{\Sigma}^J_{(n)}(\bar{z}). \tag{1.63}$$

Here $\sigma_{(n)}[k^I]$ is shorthand for the product of the twist operator σ_n and the vertex operator (1.53) that introduces momentum k^I.

In the next section we will consider string interactions that can be described by the orbifold model after adding a deformation term.

1.3 Reproducing tree level string scattering

1.3.1 Interaction vertex

The question arises, how interactions between strings can be formulated in the orbifold model (1.35). To include splitting and joining processes of strings we have to deform the model by adding an interaction term. It is clear from figure 1.7 that an interaction should somehow connect the block diagonal field configurations (1.27) of different ranks. In other words an interaction changes the quasi-periodic boundary conditions the matrix fields satisfy. Two eigenvalues of the field X_I will be interchanged at some intermediate stage, so that a group element g_1 of S_N that represents the initial string configuration is changed into another element g_2 (see figure 1.10). Then one naturally associates the group element $g = g_1^{-1} g_2$ to the interaction vertex, though of course this group element is not unique.

As an example we take the scattering process of figure 1.7. The incoming state can be represented by an S_N group element that consists of two permutations $(12\cdots n_1)(n_1{+}1\cdots N)$, whereas the intermediate string state has maximal length and therefore has an associated permutation element $(12\cdots N)$. These two group elements are related by a simple permutation

$$(12\cdots n_1)(n_1{+}1\cdots N)(n_1 N) = (12\cdots N), \tag{1.64}$$

or by any other transposition that exchanges an element of the integers $1\cdots n_1$ with another element of $n_1{+}1\cdots N$.

Hence we see that the joining process in figure 1.7 can be described by simple permutations. Likewise the splitting of one string into two strings can be described with the help of transpositions.

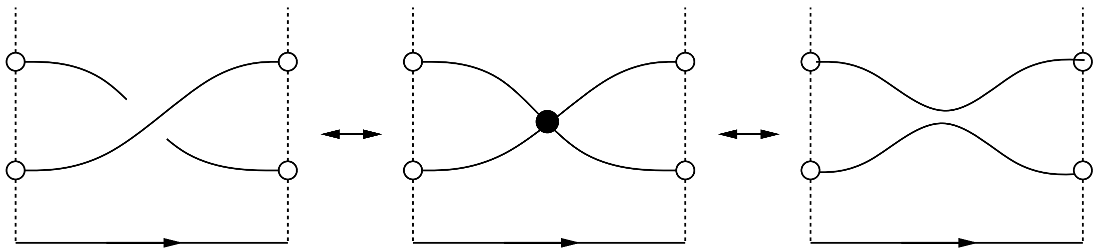

Figure 1.10 A splitting process of a long string corresponds to a change of the boundary conditions of the coordinate fields. To an elementary splitting of strings a simple permutation is associated, cf equation (1.64). Figure taken from [29].

It is quite natural to use twist fields [41] for the definition of the interaction vertex, as these twist fields change the boundary conditions of bosonic or fermionic fields. Joining and splitting of strings have a description in terms of $\mathbb{Z}_2$ twist operators, that are special cases of the operators defined in (1.51), (1.58) and (1.59). They are defined via the operator product expansions

$$\partial X_-^I(z)\cdot\sigma(w,\bar{w}) \sim (z-w)^{-\frac{1}{2}}\tau^I(w,\bar{w}), \tag{1.65}$$

$$\theta_-^\alpha(z)\cdot\Sigma_{(n)}^I(w) \sim \frac{1}{2}(z-w)^{-\frac{1}{2}}\gamma^I_{\alpha\dot{\alpha}}\Sigma^{\dot{\alpha}}(w), \tag{1.66}$$

$$\theta_-^\alpha(z)\cdot\Sigma^{\dot{\alpha}}(w) \sim \frac{1}{2}(z-w)^{-\frac{1}{2}}\gamma^I_{\alpha\dot{\alpha}}\Sigma^I(w). \tag{1.67}$$

Here the index $-$ refers to the element of $\mathbb{Z}_2$ with order 2.

With the twist fields (1.65) – (1.67) one can construct a vertex that describes the joining and splitting of strings, and that is moreover supersymmetric and manifestly

$SO(8)$-invariant [29],

$$V_{int} = \frac{\lambda N}{2\pi} \sum_{i<j} \int d^2z \left(\tau^I(z)\Sigma^I(z) \otimes \bar{\tau}^J(\bar{z})\bar{\Sigma}^J(\bar{z}) \right)_{ij}. \tag{1.68}$$

Here λ is a coupling constant of mass dimension -1, as the integrand is a weight $(\frac{3}{2}, \frac{3}{2})$ conformal field. We take λ to be proportional to the string coupling constant g_s, so that in the zero string coupling limit the interaction vertex vanishes. The vertex operator is manifestly invariant under the supercharges Q^α (1.25), as the Q^α's only depend on the zero modes θ^α_+ . The other supercharge $Q^{\dot\alpha}$ in (1.25) acts in a non-trivial way on the terms in (1.68) but leaves the vertex as a whole invariant [29]: First we note that (no summation over $\dot\alpha$)

$$\left[G^{\dot\alpha}_{-\frac{1}{2}}, \sigma\Sigma^{\dot\alpha} \right] = \frac{1}{2}\tau^I\Sigma^I, \tag{1.69}$$

where $G^{\dot\alpha}_{-\frac{1}{2}}$ is the Fourier mode with a simple pole in the expansion of the supersymmetry operator $G^{\dot\alpha}$ which has conformal weight 3/2

$$G^{\dot\alpha} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} G_{-n} z^{n-3/2}. \tag{1.70}$$

Taking the commutator with $G^{\dot\alpha}_{-\frac{1}{2}}$ on both sides of (1.69), and using the Jacobi identity we get

$$[G^{\dot\alpha}_{-\frac{1}{2}}, \tau^I\Sigma^I] = [L_{-1}, \sigma\Sigma^{\dot\alpha}] = \partial_z(\sigma\Sigma^{\dot\alpha}). \tag{1.71}$$

Because the right-hand side of (1.71) is a total derivative we conclude that the vertex operator is invariant under the supercharge $Q^{\dot\alpha}$.

The interaction vertex (1.68) is unique in the sense that it is the least irrelevant supersymmetric $SO(8)$-invariant operator we can form with the twist operators.

Long time ago Mandelstam already proposed the vertex (1.68) in the context of interactions formulated in the NSR approach to superstrings [64]. In this formulation the twist fields Σ^i play the role of the fermionic variables of the string. In the light-cone gauge the obvious geometrical three vertex function

$$\prod_{\sigma,I} \delta(X^I_{\mathrm{f}}(\sigma) - X^I_{\mathrm{i}}(\sigma)), \tag{1.72}$$

is not $SO(9,1)$ invariant. Here X_{i} and X_{f} are the string states before and after the interaction. A Lorentz invariant vertex is obtained for the NSR formalism by adding an extra term factor at the joining point [64]

$$\Sigma^I \partial X^I. \tag{1.73}$$

Remarkably when we consider the geometric joining and splitting that in the formalism of [29] takes the form of the twist operator $\sigma(0)$ and take into account the extra factor (1.73) we precisely arrive at the vertex operator (1.68)

$$\oint \frac{dz}{z^{1/2}} \Sigma^I \partial X^I(z)\sigma(0) = \tau^I\Sigma^I(0). \tag{1.74}$$

The fact that in the NSR formulation the factor (1.73) makes the three-vertex $SO(9,1)$ invariant, gives a hint that matrix string theory also has ten dimensional Lorentz invariance, although of course we expect full Lorentz invariance only in the large N limit.

1.3.2 Tree level string scattering

Now we have derived the vertex operator for elementary joining and splitting processes we can in principle calculate amplitudes of string scattering processes and compare the results with light-cone string theory.

In [4] the four graviton scattering amplitude was calculated at tree level and in [5] this result was extended to all (tree level) four particle scattering amplitudes. To compare these results with light-cone string theory, the large N limit should be taken, but it appears that for tree level amplitudes this limit is straightforward.

As the calculations of the amplitudes are rather technical we will only sketch here the general ideas, and refer to [4][5] for the details.

For a four graviton scattering process the S-matrix is up to quadratic order in λ [4]

$$\langle f|\, S\, |i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \langle f| \int d^2z_1 d^2z_2 |z_1||z_2| \mathcal{T} \sum_{i<j} V_{ij}(z_1,\bar{z}_1) \sum_{k<l} V_{kl}(z_2,\bar{z}_2)\,|i\rangle\,, \quad (1.75)$$

where V_{ij} is defined as the integrand of the integral in (1.68), $\mathcal{T}$ is the ordering operator used in radial quantization [41] and $|i\rangle$ and $|f\rangle$ are the initial and final states, each consisting of two gravitons with certain momenta and polarization,

$$|i\rangle = V_{[g_0]}[\vec{k}_1,\zeta_1;\vec{k}_2,\zeta_2](0,0)\,|0\rangle \quad (1.76)$$

and

$$\langle f| = \lim_{z\to\infty} \langle 0|\, V_{[g_\infty]}[\vec{k}_3,\zeta_3;\vec{k}_4,\zeta_4](z,\bar{z}) z^{2h}\bar{z}^{2\bar{h}}. \quad (1.77)$$

The conjugacy class $[g_0]$ can be represented as the product of two irreducible (disjunct) cycles $g_0 = (n_0)(N-n_0)$, with according to the long string picture n_0 and $N-n_0$ the p^+-momenta of the two scattering gravitons. Likewise $[g_\infty]$ has a representative $(n_\infty)(N-n_\infty)$.

The expression (1.75) gets simplified after the conformal transformation $z \to \frac{z}{z_1}$ and the introduction of a new integration variable $u = \frac{z_2}{z_1}$. The integral over z_1 contributes a delta-function of the k_- momenta, so that the S-matrix reduces to a single integral

$$\langle f|\, S\, |i\rangle = -i2\lambda^2 N^3 \delta(\sum_i k_i^-) \sum_{i<j,k<l} \int d^2u |u|\, \langle f|\, \mathcal{T}\ V_{ij}(1,1) V_{kl}(u,\bar{u})\,|i\rangle\,. \quad (1.78)$$

The expression in (1.78) involves correlation functions of four vertex functions

$$\langle V_{g_\infty}(\infty) V_{h_2}(1,1) V_{h_1}(u,\bar{u}) V_{g_0}(0,0)\rangle\,. \quad (1.79)$$

The vertex functions V_{g_∞}, V_{g_0} correspond to the twisted in- and out-states, the other two describe elementary splitting respectively joining of strings (and therefore the group elements $h_{1,2}$ are two-cycles).

The vertex functions (1.79) vanish unless the S_N group elements that label the vertex functions, satisfy the appropriate physical conditions. These conditions are determined by the particular process by which the incoming state gets transferred to the outgoing state. For the four particle scattering process we consider, we have two possibilities: One of the two incoming strings splits into two strings; out of the three intermediate strings two join together again, so that there are two outgoing strings. In the other process two incoming strings join to one string; this intermediate string propagates and splits again in two outgoing strings. To these possibilities, that are illustrated in figure 1.11, particular group elements of S_N are associated, for which the expectation value (1.79) does not vanish.

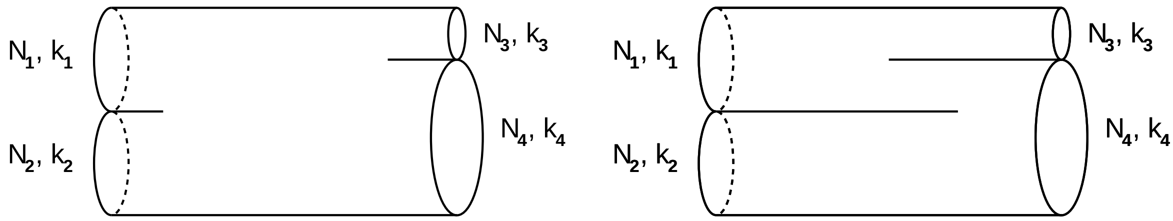

Figure 1.11 Two typical diagrams that contribute to the tree level amplitudes.

The rest of the calculation is rather tedious; for the full details we refer to the original papers [4][5]. There, firstly the correlation functions are factorized in left-moving/right-moving and bosonic/fermionic parts. Then these bosonic and fermionic correlation functions are determined explicitly, together with the appropriate normalizations.

The end result for four-graviton scattering at tree level is

$$A = \lambda^2 2^{-8} \int d^2z |z|^{\frac{1}{2}k_1 k_4 - 2} |1-z|^{\frac{1}{2}k_3 k_4 - 2} K(z, \bar{z}, \zeta), \tag{1.80}$$

where K is a kinematic factor depending on the polarizations and momenta known from tree amplitudes in string theory. The result (1.80) coincides with known results of string theory [47].

For the above result the large N limit is needed. For tree level this limit is relatively simple, because there are no subleading terms in the scattering amplitude like $1/N$ corrections. This is the case as all strings in the diagrams of figure 1.11 satisfy the standard level matching condition $L_0 - \bar{L}_0 = 0$. Note that this is no longer necessarily true for higher loop string diagrams, where the intermediate strings may *not* satisfy the usual level matching condition. We therefore expect that the large N limit will be less trivial for higher genus contributions to the scattering amplitude.

In [5] the results of [4] were further extended to all four particle scattering amplitudes of IIA string theory. They are all reproduced by the orbifold model. This gives clear evidence for the conjecture that the orbifold model is a reformulation of light-cone string theory.

It is however not entirely clear -with the knowledge reviewed in this chapter- how non-perturbative corrections may be added. Therefore we will now turn our attention to a more direct way to add the non-perturbative degrees of freedom. This approach that will be explained starting in the coming section, and in more detail in the next chapters, has the orbifold model as a special limit. Using this relation it is possible to propose a way to include the non-perturbative degrees of freedom to the orbifold model, but we will postpone this till the end of chapter three.

1.3.3 Towards matrix string theory

To get a quick understanding of matrix string theory (for a review see [30]) we will give here a somewhat heuristic introduction of the model. In the course of the next chapters we will become more precise. The ideas presented here are nevertheless important ingredients of matrix string theory, and we will need them throughout this thesis.

In principle it is possible to say in one sentence what matrix string theory is all about: matrix string theory is a supersymmetric gauge theory that contains DLCQ string theory and extra degrees of freedom that represent the non-perturbative objects in string theory. The presence of these additional degrees of freedom is the very motivation to study matrix strings: their presence gives us a way to calculate non-perturbative corrections to processes in perturbative string theory.

In the preceding sections we only did some reformulations of free strings and their interactions. The orbifold model we used for this alternative description of perturbative strings can be shown to flow out of a gauge theory in the infra-red. This gauge theory is $\mathcal{N}=8$ two dimensional supersymmetric $U(N)$ Yang-Mills theory defined on a cylinder, that can be viewed as the dimensional reduction of $\mathcal{N}=1$ SYM from ten down to two dimensions. The idea is that the matrices (1.23), which were formed out of the string coordinates in section 1.1.4, are identified as diagonal field configurations of Higgs fields in the gauge model, when the strings are widely separated. Then the differences between the eigenvalues of the Higgs fields X^I are large, so all charged fields in the SYM theory become very massive (as compared to the Higgs scalars) and they effectively decouple from the dynamics. Widely separated strings break the gauge symmetry $U(N)$ to $(U(1))^N$, but when they approach each other and/or interact, part of the broken gauge symmetry is restored. A description of interactions therefore needs the complete non-abelian theory.

The action of the gauge model is given by

$$S = \int d\tau \int d\sigma \, \mathrm{Tr}\Big\{ -\frac{g_s^2}{4} F_{\alpha\beta}^2 - \frac{1}{2}(D_\alpha X^I)^2 + \frac{1}{4 g_s^2}[X^I, X^J]^2 + i\bar{\psi}\, \not{D}\psi - \frac{1}{g_s}\bar{\psi}\Gamma^I\,[X_I, \psi]\Big\} \,. \tag{1.81}$$

We identified the Yang-Mills coupling constant with the inverse of the string coupling constant

$$g_{YM}^2 = 1/g_s^2, \tag{1.82}$$

so that the relation between the gauge model (1.81) and DLCQ string theory is an example of a strong-weak coupling duality. String interactions are perturbative processes in string theory, but non-perturbative in matrix string theory.

In equation (1.82) we wrote the Yang-Mills coupling constant (which has dimensions of inverse length) in units where the radius of the Yang-Mills cylinder is equal to one. The zero string coupling limit is therefore equivalent to the strong coupling limit and/or infra-red limit. We postpone a more concrete derivation why the gauge model (1.81) is related to DLCQ string theory, including the coupling constant identifications (1.82) till the next chapter.

The basic dynamical variables of the theory are $N \times N$ hermitian matrices and include 8 scalar fields X^I and 8 fermion fields ψ_L^a and $\psi_R^{\dot{a}}$. In the free string limit $g_s \to 0$ (or strong Yang-Mills coupling limit) the gauge fields A_α decouple from the theory and the Higgs fields satisfy the commutation relations

$$[X^I, X^J] = 0. \tag{1.83}$$

The equations (1.83) imply that the Higgs fields can be diagonalized simultaneously. They still can satisfy non-trivial boundary conditions, that are classified by the Weyl group of the gauge group. For $U(N)$ this Weyl group is the symmetric group S_N, so we are back to the orbifold model (1.35) we discussed before. We are thus lead to the conjecture that in the zero string coupling limit the gauge theory (1.81) flows to a supersymmetric conformal field theory that is defined on the orbifold (1.35).

When we relax the limit $g_s \to 0$ and take the string coupling constant to be finite, there can be non-diagonal terms in the matrix string configurations. These terms are responsible for the extra degrees of freedom present in matrix string theory as compared to light-cone string theory.

For finite g_s interactions will be included automatically, but we still have to find the appropriate solutions of the equations of motion that yield a (classical) description of joining and splitting of strings. Indeed in chapter 3 we will find an instanton-like solution that describes these elementary processes in matrix string theory. This instanton solution should be glued into a global solution that equals the asymptotic states far away from the interaction region. For a concrete comparison with string theory, the quantum fluctuations around the classical solution have to be taken into account. We will discuss this and other issues in chapter three.

Appendix A: Partition function of a symmetric product orbifold model

In [28] an identity was proven that equates the elliptic genus partition function of a supersymmetric sigma model defined on an N-fold symmetric product X^N/S_N to the partition function of a second quantized string theory on the space $X \times S^1$. Here X is a Kähler manifold. The elliptic genus of a supersymmetric sigma model is defined as the trace over the R-R sector of the evolution operator q^{L_0} times $(-1)^F y^{F_L}$. Here $F = F_L + F_R$ is the sum of left and right moving fermion number, and y is a complex parameter that counts the left moving fermion number. By virtue of supersymmetry the right moving sector contributes only via the ground states. In the special case that $y = 1$ the elliptic genus reduces to the Euler number. The orbifold Euler number can be read of from the generating functional [52][85]

$$\sum_{N\geq 0} p^N \chi(S^N X) = \prod_{n>0} \frac{1}{(1-p^n)^{\chi(X)}}. \tag{A.1}$$

The full partition function of the supersymmetric sigma model can also be related to a second quantized string theory. More concretely we claim that the following identity holds,

$$\sum_{N\geq 0} p^N \chi(S^N X; q, \bar{q}) = \prod_{\substack{n>0\\ l,m\geq 0}} \frac{1}{(1-p^n (q\bar{q})^{l/n} q^m)^{d(mn+l,l)}}. \tag{A.2}$$

Here $d(m,l)$ are the degeneracies of the single partition function, that contribute at level k and l to the operators L_0 and $\bar{L}_0$,

$$\chi(\mathcal{H}; q, \bar{q}) = \sum_{k,l} d(k,l) q^k \bar{q}^l. \tag{A.3}$$

The assumption we make is that the single Hilbert space $\mathcal{H}$ is discrete, i.e. the indices l, m in (A.3) run over a discrete set, but are not necessarily integer-valued (this in contrast to the case of the elliptic genus where all levels are integer-valued). Note that in (A.3) there are both zero mode contributions and oscillator modes contributions to the levels of q and $\bar{q}$, but we did not make this explicit in order to keep the formulas as simple as possible.

Before we prove (A.2) let us comment on the physical interpretation of the right-hand side of (A.2). The right-hand side can be interpreted as the second quantized partition function of a string, whose Fock space is made up by applying creation operators $\psi^i_{l,m,n}$, $i = 1 \ldots d(m,l)$. The parameter p counts the light-cone momentum p^+, $|q|$ and q count two quantum numbers (not necessarily integer-valued).

In the remaining part of this section we will proof (A.2), thereby heavily relying on the original proof for the case of the elliptic genus partition function [28].

In the preceding subsection we have seen that the Hilbert space of a closed string theory on a symmetric product space can be written as a direct sum of direct products of smaller Hilbert spaces, see equation (1.43). Repeatedly using the following rules for partition functions of direct sums respectively tensor products of Hilbert spaces

$$\begin{aligned} \chi(\mathcal{H} \oplus \mathcal{H}'; q, \bar{q}) &= \chi(\mathcal{H}; q, \bar{q}) + \chi(\mathcal{H}'; q, \bar{q}), \\ \chi(\mathcal{H} \otimes \mathcal{H}'; q, \bar{q}) &= \chi(\mathcal{H}; q, \bar{q}) \cdot \chi(\mathcal{H}'; q, \bar{q}), \end{aligned} \tag{A.4}$$

we get for partition function of the orbifold model

$$\chi(\mathcal{H}(S^N X); q, \bar{q}) = \sum_{\substack{N_n \\ \sum n N_n = N}} \prod_{n>0} \chi(S^{N_n} \mathcal{H}^{\mathbb{Z}_n}_{(n)}; q, \bar{q}), \tag{A.5}$$

where we used the notations of the previous subsection. To proceed we first proof that the partition function of a symmetrized tensor product of identical Hilbert spaces can be written in terms of the partition function of one single Hilbert space only. This will enable us to reduce further the righthand side of (A.5).

As already indicated we assume that the single Hilbert space $\mathcal{H}$ has a discrete spectrum. The partition function of the symmetrized tensor product of $\mathcal{H}$ is given by the generating function

$$\sum_{N\geq 0} p^N \chi(S^N\mathcal{H}; q, \bar{q}) = \prod_{k,l} \frac{1}{(1-pq^k\bar{q}^l)^{d(k,l)}}. \tag{A.6}$$

This identity can be proven as follows [28]. We define the $V_{k,l}$ to be the vector space whose dimension equals the degeneracy number $d(k,l)$ of the single Hilbert space $\mathcal{H}$. We have

$$\sum_{N=0}^{\infty} p^N \chi(S^N\mathcal{H}; q, \bar{q}) = \sum_{N=0}^{\infty} p^N \sum_{\substack{N_{k,l} \\ \sum N_{k,l}=N}} \prod_{k,l} (q^k\bar{q}^l)^{N_{k,l}} \dim(S^{N_{k,l}} V_{k,l}), \tag{A.7}$$

where $N_{k,l}$ runs over all partitions of N. The summations in the expression of the right hand side in (A.7) can easily be rewritten as

$$\sum_{N=0}^{\infty} p^N \chi(S^N\mathcal{H}; q, \bar{q}) = \prod_{k,l} \sum_{N} p^N (q^k\bar{q}^l)^N \dim S^N(V_{k,l}). \tag{A.8}$$

Finally by using the identity

$$\dim(S^N V_{k,l}) = \binom{d(k,l)+N-1}{N}, \tag{A.9}$$

we arrive at the result (A.6).

Now we return back to the goal of this section, namely calculating the partition function of the symmetric product orbifold model. The partition function of the single Hilbert space $\mathcal{H}_n^{\mathbb{Z}_n}$ is given by

$$\chi(q, \bar{q}, \mathcal{H}_n^{\mathbb{Z}_n}) = \sum_{\substack{k,l\geq 0 \\ k-l=nm}} d(k,l) q^{k/n} \bar{q}^{l/n} = \sum_{l,m} d(l+nm, l)(q\bar{q})^{l/n} q^m. \tag{A.10}$$

Combining this result with (A.8) we arrive at the result (A.2)

$$\begin{aligned}\sum_{N\geq 0} p^N \chi(S^N X; q, \bar{q}) &= \sum_{N\geq 0} p^N \sum_{\substack{N_n \\ \sum nN_n=N}} \prod_{n>0} \chi(S^{N_n}\mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, \bar{q}) \\ &= \prod_{n>0} \sum_{N\geq 0} p^{kN} \chi(S^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, \bar{q}) = \prod_{\substack{n>0 \\ l,m\geq 0}} \frac{1}{(1-p^n(q\bar{q})^{l/n}q^m)^{d(l+nm,l)}},\end{aligned} \tag{A.11}$$

where again it is understood that the labels m, l run over a discrete set, but are not necessarily integer valued. We will use the result (A.2) in chapter 3 when we calculate the degeneracies of BPS-states in supersymmetric Yang-Mills theory compactified on a three-torus.

Chapter 2

U-duality in N=4 Yang-Mills theory on T^3

Toroidal compactified type II string theory is conjectured to be invariant under a discrete symmetry group, called U-duality [59]. Relations between gauge models and string theory suggest that this string duality should be reflected in gauge theories as well. In this chapter we will review why this is indeed the case. We will moreover show that gauge theories know about extended U-duality symmetries. That is, certain properties of the theories, like BPS mass spectra, or BPS state degeneracies, can be shown to be invariant under a larger symmetry group, than one might at first sight expect.

In the gauge theory interpretation this extended symmetry group is a combination of electro-magnetic duality, the mapping class group of tori, and Nahm-type dualities.

We will mainly concentrate on a supersymmetric Yang-Mills model defined on a three-torus. We will study in detail the degeneracies of BPS states in this model, and show that they exhibit a U-duality symmetry. In the last section we explain in detail the appearance of the duality symmetry, and its absence in the BPS mass spectrum.

First, however, we will introduce some basic concepts. We start with a brief explanation of string duality. Then we give a review of D-branes, and their low energy description. After this we introduce M-theory, a conjectured eleven-dimensional model that unifies all known string theories. The notions of D-branes and M-theory naturally lead us to the matrix theory proposal of [9], which says that M-theory in a special regime has a particularly simple description in terms of a matrix quantum mechanics. Matrix string theory is closely related to this model, and following [76] we will give arguments why it yields a description of non-perturbative IIA strings.

2.1 D-branes and string dualities

2.1.1 String duality

First we will briefly discuss string duality in closed string theory. The bosonic massless modes of closed string theory come from two sectors, the RR sector and the NS-NS sector:

$$\begin{array}{lll} \mathrm{NS-NS} & |\mu>_L \otimes |\nu>_R \longrightarrow & G_{\mu\nu}, B_{\mu\nu}, \phi \\ \mathrm{R-R} & |\alpha>_L \otimes |\beta>_R \longrightarrow & \text{antisymmetric forms } A^i \end{array} \tag{2.1}$$

Here G is the ten-dimensional space-time metric, B an antisymmetric two form, ϕ the dilaton field and the A^i are antisymmetric RR forms. The RR gauge fields that satisfy definite chirality conditions, form together with the NS-NS fields the ground states of either type IIA or IIB string theory. This is illustrated in the next table

	NS − NS	R − R
IIA	$G_{\mu\nu}, \phi, B_{\mu\nu}$	A^1, A^3
IIB	$G_{\mu\nu}, \phi, B_{\mu\nu}$	A^0, A^2, A^4

Table 2.1 The bosonic massless fields of type IIA and IIB string theory

The low energy dynamics of these massless modes have a field theory description in terms of supergravity theories.

A remarkable property of these field theories are the so-called duality symmetries. These classical symmetries are expected to be quantum mechanical symmetries in string theory. They come in two types: T-duality and S-duality. A single T-duality transformation (for a review of T(arget space)-duality see [43]) changes the chirality of the theory from type IIA to type IIB [22][31] and in the opposite direction. In its simplest non-trivial form it inverts the radius R of a compact direction to α'/R, and exchanges the momentum and winding modes of strings winded along the compact direction, thereby leaving the total mass spectrum invariant. The RR forms get an extra index or lose one, depending on whether the index was already there or not. It thus changes the rank of the RR tensor fields by ± 1, so that we get a map from type IIA string theory to IIB and vice versa. Together with the mapping that acts by integral shifts in the fields, the $R \to \alpha'/R$ duality forms an $SL(2,\mathbb{Z})$ symmetry group. Generalized to d compactified dimensions the T-duality group becomes $SO(d,d)$. For a detailed account of the action of T duality on the fields we refer to [16] and [43].

D	d	Sugra	string theory
10	1		
9	2	$SL(2,\mathbb{R})$	$SL(2,\mathbb{Z})$
8	3	$SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$	$SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$
7	4	$SL(5,\mathbb{R})$	$SL(5,\mathbb{Z})$
6	5	$SO(5,5,\mathbb{R})$	$SO(5,5,\mathbb{Z})$
5	6	$E_6(\mathbb{R})$	$E_6(\mathbb{Z})$

Table 2.2 The U-duality groups of type II string theory in different dimensions

Ten dimensional type IIB string theory is conjectured to have S-duality as a genuine symmetry [59]. It is a non-perturbative $SL(2,\mathbb{Z})$ symmetry; it relates the weak coupling regime to the strong coupling regime, and it is therefore reminiscent of electromagnetic duality in gauge theories. The dilaton forms together with the RR-field A^0 a complex scalar that transforms under fractional $SL(2,\mathbb{Z})$ transformations. The two-forms B and A^2 transform as a doublet.

The generators of both types of duality groups do not commute, together they form a larger group that goes under the name of U-duality [59]. The U-duality groups become larger in lower dimensions and are discrete: As indicated in table 2.2, in eight dimensions the symmetry group is $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$, in seven $SL(5,\mathbb{Z})$ and in six dimensions $SO(5,5,\mathbb{Z})$.

2.1.2 D-BRANES

An immediate corollary of the conjecture that IIB string theory has S-duality invariance is the existence of a dual string that transforms together with the elementary string in a doublet. This dual string has been mysterious for a long time, until Polchinski realized that the D-string could play this role [70]. A D-string is a particular example of $p+1$ dimensional D-branes, which were previously known as hyper-surfaces with the property that open strings can end on them [22]. Polchinski also clarified the role of the RR-fields by noting that they should couple to the D-branes [70].

One explains the possibility that strings can end on a hyper-surface by considering the boundary conditions that the coordinate fields of open strings have to satisfy at their endpoints. There are two types of possible boundary conditions for the open string,

$$\begin{aligned} \partial_\perp X^\mu &= 0 \quad \text{(Neumann)}, \\ \delta X^\mu &= 0 \quad \text{(Dirichlet)}. \end{aligned} \tag{2.2}$$

Dirichlet boundary conditions break Poincaré invariance, and hence represent topological defects in space-time. These topological defects are called D-branes, which is shorthand for Dirichlet branes. A static D-brane with p spatial dimensions is described by the boundary conditions

$$\partial_\perp X^{0,1,\ldots,p} = 0, \qquad X^{p+1,\ldots,9} = 0. \tag{2.3}$$

The massless modes of the open string tied to the world-volume of the brane have a description in terms of a low energy field theory on the D-brane. Thus we get a field $A^\mu(\xi^\alpha)$, where ξ^α is a parameterization of the world-volume, that decomposes into a parallel gauge field $A^\alpha(\xi^\alpha)$ on the brane and transversal components $X^I(\xi^\alpha)$ that form scalar fields.

The effective action of a (fluctuating) D-brane was originally derived by requiring that its equations of motion reproduce the conditions that are implied by conformal invariance of open strings in the D-brane background [22][63]. Another, less technical, way to derive the action is by noting that the D-brane low energy effective action is essentially fixed by Lorentz invariance and T-duality [7]. To see this we start with the example of a D-particle moving through flat spacetime [7]. We take its world-line through ten-dimensional space-time parameterized in the following way

$$X^0(\tau) = \tau \qquad X^I = X^I(\tau). \tag{2.4}$$

The point particle Lagrangian is

$$S = \tau_0 \int d\tau \sqrt{1 - (\partial_0 X_I)^2}. \tag{2.5}$$

The action (2.5) is an effective action in the sense that higher order derivatives with respect to time, e.g. acceleration terms, are neglected. We have put the RR fields to zero, which would otherwise couple to the D-particle through Wess-Zumino terms.

By a T-duality transformation in the 1 direction the boundary condition of the open string coordinate field X^1 turns from Neumann to Dirichlet, the D-brane scalar field X^1 becomes a gauge field A^1 and hence the D-particle becomes a D-string. The velocity of the D-particle in the 1 direction plays the role of a field strength on the D-string

$$F_{01} = \partial_0 A_1 = \dot{X}_1/2\pi\alpha'. \tag{2.6}$$

So we get for the effective world-volume action of the D-string

$$S = \tau_1 \int d^2\xi \sqrt{1 - (\partial_0 X_I)^2 - (2\pi\alpha' F_{01})^2}, \tag{2.7}$$

where I now runs from $2 \ldots 9$. The same argument can be repeated for other directions, so that we get higher dimensional D-branes. This leads us to the form of the bosonic low energy effective action of a $p+1$ dimensional D-brane

$$S = T_p \int d^{p+1}\xi \, e^{-\phi} \sqrt{\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}. \tag{2.8}$$

Here T_p is the brane tension. We included the pull-back of the antisymmetric NS-NS $B_{\mu\nu}$-field. Together with the pull-back of the RR field strength it forms the $U(1)$ gauge invariant world-volume field strength $2\pi\alpha'\mathcal{F}_{\alpha\beta} = B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}$ that lives on the D-brane world-volume. The metric $\hat{G}$ is the pullback of the space-time metric to the D-brane volume

$$\hat{G}_{\alpha\beta} = G^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu. \tag{2.9}$$

In addition to the action (2.8) there are couplings of the D-brane to RR fields in the form of Wess Zumino terms.

The Born-Infeld action (2.8) as an effective description of D-branes can also be verified by a perturbative string calculation [71]. This open string calculation moreover gives the actual value of the brane tension T_p; the result is [71]

$$\tau_p = \frac{T_p}{g_s} = \frac{1}{g_s\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p}, \tag{2.10}$$

where the string coupling g_s is related to expectation value of the dilaton $g_s = e^{<\phi>}$. As expected the brane tension is proportional to the inverse of the string coupling constant, so that the D-branes are indeed non-perturbative objects.

With some further restricting assumptions the form of the BI action (2.8) simplifies considerably: in the zero-slope limit $\alpha' \to 0$ combined with additional restrictions, the BI action reduces to abelian Yang-Mills theory.

To show this we take for simplicity the background metric and the D-brane to be flat. The pullback of the metric to the brane is then

$$G_{\alpha\beta} = \eta_{\alpha\beta} + \partial_\alpha X^I \partial_\beta X^I + \dots . \tag{2.11}$$

Furthermore, we set the antisymmetric tensor B to zero, and assume that the terms $2\pi\alpha' F_{\alpha\beta}$ and $\partial_\alpha X^I$ are small. Then we can expand the Born-Infeld action as follows

$$S = T_p V_p + \frac{1}{4g_{\mathrm{YM}}^2} \int d^{p+1}\xi \left(F_{\alpha\beta}^2 + \frac{2}{(2\pi\alpha')^2} \partial_\alpha X^I \partial_\beta X^I \right) + \dots , \tag{2.12}$$

where we made the identification

$$g_{\mathrm{YM}}^2 = \frac{1}{4\pi^2(\alpha')^2 \tau_p}. \tag{2.13}$$

The action (2.12) is the bosonic part of 10-dimensional $\mathcal{N} = 1$ SYM, dimensionally reduced to $p+1$ dimensions. Though this theory is a *truncated* model of D-branes, it has some peculiar properties in favor when compared with Born-Infeld theory. One can easily add fermionic degrees of freedom, and especially one can replace in a straightforward way the abelian gauge group by non-abelian $U(N)$ groups. This extension to non-abelian groups is useful, because when we have more than one D-brane, the dynamics get naturally a description in terms of $U(N)$ gauge theory.

Thus D-branes shed new light on the intimate relation between string theory and (non-abelian) gauge theories. We will explain this in more detail in the next section.

2.1.3 Bound states of N D-branes

In the simplest case of two parallel D-branes there are two hyper-surfaces on which open strings can end. These possibilities can be included in string theory by adorning the open strings with extra degrees of freedom at their endpoints, called Chan Paton factors, that simply indicate which brane the endpoint is restricted to. With the inclusion of these Chan Paton factors, string wave functions have decompositions like

$$|k; a\rangle = |k; ij\rangle \, \lambda_{ij}^a, \tag{2.14}$$

where the λ_{ij}^a are, in this case, 2×2 matrices that label the Chan Paton factors. The matrices λ_{ij}^a are allowed to be simultaneously conjugated by $U(2)$ transformations, as in string amplitudes only traces of products of Chan Paton factors appear.[1]

[1] We consider oriented open strings. For non-oriented strings the gauge group changes to $SO(N)$ or $USp(N)$.

On each of the two D-branes we have a $U(1)$ gauge field that couples to open strings. These fields can be thought of as two abelian directions in a larger $U(2)$ group. This $U(2)$ gauge group is equal to $U(1)\times SU(2)/\mathbb{Z}_2$. The $U(1)$ subgroup describes the center of mass motion of the branes, while the non-abelian $SU(2)$ part determines the relative motion. The Weyl group $\mathbb{Z}_2$ of $SU(2)$ acts by permuting the branes, corresponding to the fact that they are indistinguishable.

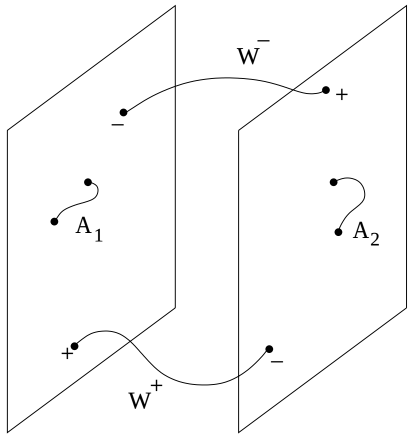

Figure 2.1 The low energy dynamics of two separated branes has a description in terms of a $U(1)\times U(1)$ broken sector of $U(2)$ gauge theory. The open strings that begin on one brane and end on another, are coupled to massive charged W-bosons. The open strings beginning and starting on the same brane couple to the abelian $U(1)$ gauge fields on the brane. When the branes are on top of each other the complete gauge symmetry gets restored [86].

When the branes are separated the $SU(2)$ gauge group is broken to $U(1)$

$$U(1)\times SU(2)\longrightarrow U(1)\times U(1), \tag{2.15}$$

much like in spontaneously broken gauge theories. For example the role of the charged $W^{\pm}$ bosons is played by the ground states of strings beginning on one brane and ending on another brane, and the masses of the $W^{\pm}$ particles are proportional to the distance between the D-branes, and thus vanish when they are on top of each other (in this case the complete gauge group $U(2)$ gets restored). Thus spontaneous symmetry breaking in gauge theory can be visualized in string theory by pulling D-branes apart. This is an example of a technique which is called geometric engineering, which can be used to give phenomena in gauge theories a geometric interpretation.

In case of N parallel D-branes the total gauge symmetry group is $U(N)$, which is broken to a subgroup, depending on the relative locations of the separate D-branes. The effective description of N D-branes has an obvious generalization in terms of non-abelian Yang-Mills theory. Also the fermionic degrees of freedom can be included in a straightforward way. In this way we arrive at ten-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory, dimensionally reduced to the dimension of the D-branes, which has precisely the right field content.

As noted above this Yang-Mills description of D-branes is only valid in the zero-slope limit $\alpha'\to 0$. When we add up all α' corrections we would expect to arrive at a generalization of Born-Infeld theory, like in the abelian case. The generalization of Born-Infeld theory to non-abelian gauge groups with or without inclusion of fermionic degrees of freedom is however not yet fully established. [2] There have been proposals for a non-abelian generalization [84]. By considering the equations of motion implied by the action, the right one can be singled out almost uniquely. There appears, however, to be an ambiguity in the choice of representation that is to be used for the trace over the gauge group. For a discussion of these matters see [84].

2.1.4 Bound states within D-branes

In supersymmetric gauge theories electric charged particles can combine with magnetic charged solitons to dyonic bound states. These states usually break part of the supersymmetry and saturate a BPS mass bound. Their masses are smaller than the added masses of single electric and magnetic objects; in other words the binding energy is non-zero and the dyons are therefore truly bound.

In a quite analogous way D-branes can form bound states with p-branes (extended fundamental objects in string theory) in type IIB string theory. As an example we take bound states of fundamental strings (F-strings) and D-strings. These two types of strings transform as a doublet under the $SL(2,\mathbb{Z})$ action of S-duality, so string duality predicts the existence of a whole tower of bound states of p F-strings and q D-strings, with p and q relatively prime. It also suggests the following binding tension formula for a (p,q) string [75]

$$\tau_{p,q} = \frac{1}{2\pi\alpha'}\sqrt{p^2 + \frac{q^2}{g_s^2}}. \tag{2.16}$$

This formula implies that for weak coupling the mass difference between a single F-string plus a single D-string, and a $(1,1)$ bound state is equal to the mass of a single fundamental string up to zeroth order of the coupling constant. Apparently the F-string dissolves almost entirely in the D-string when they form a bound state by minimizing their energy. This can be visualized by imagining the following procedure [87]. Consider a fundamental string (F-string) in type IIB wrapped around a compact dimension. In the absence of D-strings, the string winding number is conserved. This winding number is equal to the conserved charge of the gauge field that can be formed out of the NS-NS two form B by integrating it over the compact direction [72][47]. In the presence of a D-string however the winding number of an F-string is no longer conserved. Instead the charge of the gauge field obtained from the $U(1)$ field strength $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + B_{\mu\nu}/2\pi\alpha'$ on the D-string is conserved. Now the closed F-string can break in two parts and "disappear", thereby leaving electric flux on the world-volume of the D-string behind, in such a way that the charge of the abelian gauge field on the D-string remains conserved.

[2] The appropriate supersymmetric version of the Born-Infeld action, that describes both the fermionic degrees of freedom and the bosonic degrees of freedom has been constructed for the abelian case [1].

A D-string with electric flux on it is thus interpreted as a bound state of a D-string and an F-string. It is a BPS state, because the configuration can be mapped to a D-particle with non-zero momentum via T-duality, cf. (2.5)-(2.7).

A proof that the bound states really exist has been given by Witten. In [87] he argued that the existence of a D-string/F-string bound state can be proven by considering the relevant low energy effective theory, perturbed with a mass term. This mass term will break part of the supersymmetry but it does not affect the BPS mass. By tuning the mass parameter of the extra term in the supersymmetric Yang-Mills theory, the effective coupling constant can be made small. For this theory then, it can be shown that the relevant supersymmetric ground state exists (and is unique) [87]. The existence and uniqueness of the ground state is not affected by putting the mass term to zero again, so that the whole tower of (p, q) strings states with p and q relatively prime do exist in string theory.

Analogous results hold for gauge theories that are associated with higher dimensional branes. By turning on electric and magnetic fluxes not only the D-brane itself is described but also bound states with lower dimensional branes and strings.

YM flux	String interpretation
Rank N	# Max. dim D-branes
Electric flux	F-string winding number
Magnetic flux	# Codim 2 D-branes
Instanton number	# Codim 4 D-branes

Table 2.3 Bound states of a D-brane with lower dimensional branes and fundamental strings have a low energy description in gauge theory. This table contains the translation code for the two theories.

This is illustrated in table 2.3. We will come back to this in section 2.3 .

2.1.5 M-THEORY AND IIA STRING THEORY

M-theory is a conjectured model in eleven dimensions that should unify all known string theories, and whose low energy limit is eleven-dimensional supergravity.

A concrete formulation of the model has not been established at present, but there are some proposals for particular sectors of the theory. A definition of M-theory could be the strong coupling limit of type IIA string theory. The motivation for this definition is, among other facts, the presence of D-particles in IIA theory whose masses depend on the inverse of the string coupling constant g_s. When they are viewed as Kaluza Klein states of an eleven dimensional theory, the radius of the extra eleventh dimension should be proportional to g_s

$$R_{11} = g_s l_s. \tag{2.17}$$

Then in the strong coupling limit an extra dimension in IIA string theory appears. Another motivation is the already long known fact that the dimensional reduction of 11D supergravity theory on a circle is IIA supergravity, the effective field theory of 10D type IIA string theory. The bosonic field content of 11D Sugra [21] is the eleven dimensional metric g_{MN} and an antisymmetric three-form gauge potential C_{MNP}. Upon dimensional reduction we therefore get a scalar, a gauge field $g_{\mu 11}$, a two form $C_{\mu\nu 11}$, a ten-dimensional metric and a three form, precisely the massless bosonic fields of IIA supergravity theory.

A graviton with momentum N/R_{11} in the eleventh direction gets the interpretation as a bound state of N D-particles, whose masses add up to $m = N/R_{11}$. The Kaluza Klein mode of the graviton $g_{\mu 11}$ couples to these D-particles and gets the role of the RR gauge field. The membrane, an extended solitonic object in 11D supergravity, wrapped along the eleventh direction gets associated to the IIA elementary string (as was realized for the first time in [83].)

M-theory	**IIA String Theory**
11D graviton	D particle
wrapped membrane	fundamental string
membrane	membrane
wrapped five brane	D four brane
five brane	NS five brane
KK gauge field	RR gauge field

Table 2.4 The dictionary between M theory (11D supergravity) compactified on a circle and IIA string theory.

The other solitonic object in supergravity, the five-brane, gives rise to the D four-brane and NS five brane. These identifications and others are summarized in the table 2.4.

One can also read off the spectrum of 11D Sugra and M-theory from the eleven dimensional superalgebra [11]

$$\{Q_\alpha, Q_\beta\} = 2p^\mu \gamma_{\mu\alpha\beta} + 2Z^{\mu_1\mu_2}\gamma_{\mu_1\mu_2\alpha\beta} + 2Z^{\mu_1\cdots\mu_5}\gamma_{\mu_1\cdots\mu_5\alpha\beta}, \tag{2.18}$$

where $\mu_i = 0, \cdots 10$ and $\alpha, \beta = 1, \cdots 32$. The superalgebra (2.18) includes two central charges: a two brane charge $Z^{\mu_1\mu_2}$ and a five brane charge $Z^{\mu_1\cdots\mu_5}$. These charges are infinite in non-compact eleven dimensional space, but in compact space they can have finite values.

The superalgebra (2.18) is well known to have U-duality invariance. The continuous versions of the U-duality groups are classical symmetries of the action of dimensional reduced eleven dimensional supergravity. The symmetries can be thought of as being generated by two groups, namely the T-duality group and the mapping class group of the torus on which 11D Sugra (M theory) should be compactified to get supergravity theories (type II string theory) in lower dimensions. The U-duality groups can thus be viewed as $SO(d-1,d-1,\mathbb{Z}) \bowtie SL(d,\mathbb{Z})$. We list the groups in different dimensions in table 2.2.

2.2 Matrix Theory

2.2.1 The matrix theory proposal

In the previous section we saw that D-particles are the only objects that carry p_{11} momentum in M-theory. This was an important motivation for the following conjecture due to Banks, Fischler, Shenker and Susskind [9]

> **Conjecture [9]**: *M-theory in the infinite momentum frame is exactly described by the $N \to \infty$ limit of 0-brane quantum mechanics*
>
> $$S = \frac{1}{2R_{11}} \int dt \,\mathrm{tr} \left(\dot{X}^2 + [X^I, X^J]^2 + \theta^T (i\dot{\theta} - \Gamma_I [X^I, \theta]) \right), \tag{2.19}$$
>
> *where N/R_{11} plays the role of the 11D momentum, and where N/R_{11} and R_{11} are both taken to ∞.*

In writing down the D-particle action (2.19) we used units where $2\pi\alpha' = 1$. The matrices X are elements of the gauge group $U(N)$, and R_{11} is the radius on which M theory should be compactified to get type IIA string theory. When the fields X are large, the finite energy configurations lie in the flat directions. Along flat directions the fields X^I are simultaneously diagonalizable and thus it is possible to interpret these diagonal matrix elements as coordinates of D-particles, with kinetic energy $M_{D0}\dot{X}^2/2 = \dot{X}^2/2R_{11}$. For small distances the fields no longer necessarily commute and the interpretation of the eigenvalues of the matrices as coordinates gets obscured. The off-diagonal components describe strings stretched between the branes, with characteristic energies $\sim |x_i - x_j| R_{11} M_{Pl}^3$, where M_{Pl} is the eleven dimensional Planck mass. The action neglects string oscillations and higher energy excitations (for example brane creation).

Matrix theory has properties of a theory in the infinite momentum frame. It is an effective theory of D-particles, so by construction it has only states with positive momentum in the eleventh direction. Furthermore the model (2.19) has ten-dimensional (super)-Galilean invariance. Because of this it seems plausible that in the infinite momentum frame of M-theory the states are primarily composed of D-particles, whose effective low energy dynamics is determined by (2.19).

Other important evidence for the model (2.19) is the natural appearance of supermembranes in matrix quantum mechanics, as was originally observed in [24]. The matrix theory Hamiltonian is exactly the same as the light-cone Hamiltonian of the supermembrane [24]. Due to supersymmetry this Hamiltonian has a continuous spectrum [25], which from the membrane point of view may be disappointing (as this would seem to rule out a generalization of strings in terms of membranes). Matrix theory, however, gives a new interpretation of this result. The continuous supermembrane spectrum belongs to the collective dynamics of (many) D-particles. Then the continuity of the spectrum is exactly what we want.

Additional evidence for the model (2.19) came from calculations of D-particle scattering amplitudes, whose results are to be compared with supergravity [13][14][73]. The first calculation, presented in [9] was the remark that in the Born approximation the scattering amplitude of two gravitons in 11D supergravity corresponds with the leading potential term between two D-particles. In [13] this agreement was checked, up to a two loop calculation in matrix theory.

One can also investigate in how far the symmetries of M-theory are present in matrix theory. These symmetries are Lorentz symmetry and U-duality symmetries. We will comment on this in the coming sections.

In calculations the rank N is usually taken to be fixed and large. For finite N the model (2.19) has been conjectured to describe M-theory in the DLCQ formalism [80]. One might view this as a stronger conjecture than the original one in [9], as it tells us something about the matrix quantum mechanics for any rank N. But the large N limit needed in the conjecture of [9] is a rather subtle issue and should be considered with great care.

We will go on with the proposal of [80] in the next section, and adapt it to matrix string theory. In particular we show how the close relation between matrix string theory and DLCQ string theory can be made plausible.

2.2.2 The matrix string theory proposal

We can now be more precise on the relation between string theory and matrix string theory: the matrix string theory proposal at finite N is type IIA string theory in the DLCQ formalism, with rank N and p^+ momentum identified. The D-particle number in string theory has the interpretation of (integer) electric flux in matrix string theory.

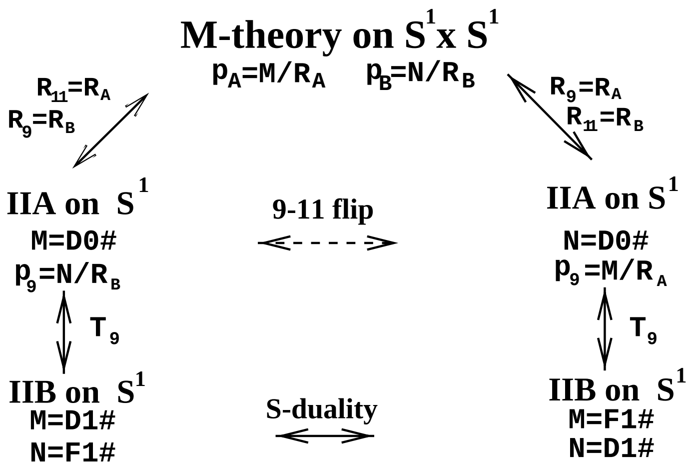

Figure 2.2 This diagram illustrates the 9-11 flip which is one of the essential features of matrix string theory. All identifications only depend on the M-theory conjecture [86] and string duality.

We can make this claim plausible by the following argument [76]. In figure 2.2 we consider M-theory compactified on two circles with radii R_A and R_B. Either of these radii can be chosen to be in the eleventh direction, so we can define two copies of type IIA string theories compactified on a circle of radius R_A respectively R_B.

The string coupling and string scale are determined by the eleventh radius and the 11-dimensional Planck scale l_{Pl} through

$$R_{11} = g_s l_s = g_s^{2/3} l_{Pl}, \tag{2.20}$$

so that the type IIA string theory obtained via compactification of the A direction has string coupling and string scale given by

$$g_s = (R_A/l_{Pl})^{3/2}, \qquad \alpha' = R_A^{-1} l_{Pl}^3. \tag{2.21}$$

By a further T-duality in the nine direction on both sides of diagram 2.2 we get two type IIB theories that are related by an S-duality transformation. The type IIB theory in the lower-right corner in figure 2.2 is defined on a circle of radius R_A' and has string coupling and string scale (see [16] for a translation code between IIA and IIB)

$$R_A' = l_{Pl}^3 R_A^{-1} R_B^{-1}, \qquad \tilde{g}_s = R_B/R_A, \qquad \tilde{\alpha}' = R_B^{-1} l_{Pl}^3. \tag{2.22}$$

The idea now is to obtain a relation between DLCQ IIA string theory and Yang-Mills theory by an infinite boost in the R_B direction of the equivalent IIA and IIB string theories (2.21) and (2.22). This infinite boost is accompanied by a zero size limit of radius R_B such that

$$R_B = e^{-\chi} R \quad (\chi \to \infty), \tag{2.23}$$

where χ is the boosting parameter which we will take to be $\cosh\chi = (R^2+4R_B^2)^{1/2}/2R_B$. This procedure has the effect that one of the light-like coordinates $x^{\pm} = x_B \pm t$ becomes compactified. Namely, for finite χ we have the simultaneous identifications

$$x^+ \simeq x^+ + e^{-\chi} R \quad , \quad x^- \simeq x^- + R \quad (\chi \to \infty), \tag{2.24}$$

so that in the limit $R_B \to 0$ the light-cone coordinate x^- becomes compact, while x^+ is identified with time. In this way we get on the left in figure 2.2, DLCQ IIA string theory in the sector $p^+ = N/R$ and D-particle number M, with finite string coupling and string scale given by (2.21).

On the right we have weakly coupled type IIB theory with N D-strings and M F-strings wrapped along a circle of large radius

$$R_A' = l_p^3 R_A^{-1} R_B^{-1} = e^{\chi}\, l_p^3 R_A^{-1} R^{-1} \quad (\chi \to \infty). \tag{2.25}$$

At the scale of this compactification radius, the string length $\tilde{\alpha}'$ vanishes

$$R_A'^2/\tilde{\alpha}' = e^{\chi}\, l_p^3 R_A^{-2} R^{-1} \quad (\chi \to \infty). \tag{2.26}$$

This means that the IIB theory gets a Yang-Mills theory description, which can be derived from the low energy effective D-string action (Born-Infeld theory) via the zero slope limit $\alpha' \to 0$. Using (2.13) and (2.22) we find the Yang-Mills coupling constant of the gauge theory description

$$g_{YM}^2 = \frac{g_s'}{\tilde{\alpha}'} = \frac{R_B^2}{R_A l_p^3}. \tag{2.27}$$

In this formula g_s' is the string coupling constant of the type IIB theory on the right side of figure 2.2.

There is one additional condition for this reduction to make sense: namely finiteness of the Yang-Mills energy, as compared with the energy scale determined by the compactification radius R_A'.

The relevant Yang-Mills Hamiltonian is the energy of a sector with integer electric flux M,

$$H_{YM} = g_{YM}^2 R_A' \frac{M^2}{2N}. \tag{2.28}$$

This follows from the remarks made in section 2.1.4 where we explained that a (p,q) string bound state has a low energy description in terms of Yang-Mills theory with electric flux turned on. The Yang-Mills energy (2.28) is finite with respect to the Yang Mills radius R_A', as can be verified by using (2.25) and (2.27).

When we include transversal momenta $p_\perp$ of the strings the YM Hamiltonian becomes

$$H_{YM} = \frac{1}{2p^+}(p_\perp^2 + \frac{M^2}{g_s^2}), \tag{2.29}$$

which equals the DLCQ string theory Hamiltonian with D-particle charge M [80].

We come to the conclusion that DLCQ IIA string theory is equivalent to super Yang-Mills theory defined on a cylinder, with dimensionless coupling constants identified via

$$g_{YM} = \frac{1}{g_s}. \tag{2.30}$$

The equivalence is thus an example of a strong-weak coupling duality as already emphasized in chapter 1.

The relation between matrix string theory and M-theory in the discrete light-cone gauge is reflected at the level of the superalgebras too. The superalgebra of DLCQ 11-dimensional supergravity (2.18) on a d-dimensional torus is identical to the one of maximally supersymmetric Yang-Mills theory on the dual torus, after appropriate identifications of the fluxes [11].

2.2.3 (De)compactification

In the previous section we argued why matrix string theory yields a dual description of non-perturbative type IIA string theory. The gauge model describes type IIA strings in the DLCQ formalism, moving in eight transversal dimensions.

The way of reasoning of the previous section can be repeated for type IIA string theory toroidally compactified to lower dimensions. The result is that the dual supersymmetric $U(N)$ gauge model is defined on a higher dimensional torus. This procedure, however, is of limited use, as Yang-Mills theories are quantum mechanically well defined only up to 4 dimensions.

This is one of the reasons why we will focus on the well known $\mathcal{N} = 4$ $U(N)$ SYM model in 3+1 dimensions, which, according to the reasoning above, describes three-branes or DLCQ string theory in 8 dimensions.

Instead of repeating this reasoning again we will show that dimensional reduction of the gauge theory means decompactification of the associated string theory, and vice versa. To begin with let us consider the Lagrangian of the $\mathcal{N} = 4$ model

$$\begin{aligned} S &= -\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(D_\mu X^I)^2 + \psi\Gamma^i D_i\psi \\ &\qquad + \psi\Gamma^I[X^I,\psi] + \frac{1}{4}[X^I,X^J]^2\Big), \end{aligned} \tag{2.31}$$

with $I = 1,\ldots 6$. When we compactify this model on a three torus of the form $T^2\times S^1$, and rescale the circle according to $x^3 \to \lambda x^3$, the bosonic part of the action (2.31) becomes

$$S = -\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\frac{1}{2\lambda}F_{3i}^2 + \frac{1}{4}\lambda F_{ij}^2 + \frac{1}{2\lambda}(D_3X^I)^2 + \frac{1}{2}\lambda(D_iX^I)^2 + \lambda[X^I,X^J]^2\Big). \tag{2.32}$$

In the limit $\lambda\to\infty$ (2.32) is conjectured to flow to an effective supersymmetric conformal field theory in the infrared [51][29]. In this limit finite energy configurations satisfy the flatness conditions

$$F_{ij} = 0, \qquad D_iX^I = 0, \qquad [X^I,X^J] = 0. \tag{2.33}$$

These conditions are the same flatness conditions one gets in the strong coupling limit of matrix string theory (see also section 1.2). We thus see that the limit $\lambda\to\infty$ (or equivalently shrinking the transversal two torus to zero size) dimensionally reduces the model (2.31) to $1+1$ dimensions, i.e. to matrix string theory. The associated string theory is however decompactified from six to eight transversal dimensions.

Conversely compactifying string theory to lower dimensions lead to higher dimensional gauge theories. Note that we have to define what we actually mean by compactifying matrix string theory. Hereto we will follow the original argument due to Taylor [81], who discussed toroidal compactifications of matrix theory.

Matrix string theory is equivalent to the low-energy description of a system of N D-strings moving in flat space $\mathbb{R}^9$. An obvious definition of compactified matrix string theory is therefore the low energy description of D-strings living on compact space.

A collection of D-strings on a circle can be described by using orbifold techniques. The covering space of a circle S^1 is the real line $\mathbb{R}$, so we can study the D-strings by considering the motion of an infinite family of D-strings on $\mathbb{R}$ and then impose constraints that imply translation invariance of the D-string configuration. The matrices

X^I get two type of indices: $X^I_{mi,nj}$ where $n \in \mathbb{Z}$ and i runs over integers $1, \cdots, N$. The conditions the discrete symmetry $\mathbb{Z}$ imposes on these fields read [81][82]

$$\begin{array}{rcll} X^I_{mn} & = & X^I_{(m-1)(n-1)} & I < 9, \\ X^9_{mn} & = & X^9_{(m-1)(n-1)} & m \neq n, \\ X^9_{nn} & = & X^9_{(n-1)(n-1)} + 2\pi R_9 \mathbb{1}, & \end{array} \tag{2.34}$$

where we suppressed the indices i, j of the matrices. We took the compactified direction in the ninth direction. As a result of the constraints (2.34) the matrix X^9_{mn} can be written in the following form [82]

$$X^9 = \begin{pmatrix} \ddots & X_1 & X_2 & X_3 & \ddots \\ X_{-1} & X_0 - 2\pi R_9 \mathbb{1} & X_1 & X_2 & X_3 \\ X_{-2} & X_{-1} & X_0 & X_1 & X_2 \\ X_{-3} & X_{-2} & X_{-1} & X_0 + 2\pi R_9 \mathbb{1} & X_1 \\ \ddots & X_{-3} & X_{-2} & X_{-1} & \ddots \end{pmatrix}, \tag{2.35}$$

where we used the shorthand notation $X_k = X^9_{0k}$. One can interpret the matrix of the form (2.35) as a representation of the covariant derivative of YM theory defined on the dual circle

$$X^9 = i\hat{\partial} + A(\hat{x}), \tag{2.36}$$

acting on the Fourier components of a periodic function defined on the dual circle

$$\phi(\hat{x}) = \sum_n \hat{\phi}_n e^{in\hat{x}/\hat{R}_9}. \tag{2.37}$$

Decomposing the gauge field $A(\hat{x})$ into Fourier components

$$A(\hat{x}) = \sum_n A_n e^{in\hat{x}/\hat{R}_9}, \tag{2.38}$$

we see that the covariant derivative (2.36) has precisely the same action on the Fourier components of the function (2.37) as the matrix (2.35). Thus we conclude that compactification of matrix string theory on a circle means adding an additional compact dimension to the YM theory, plus replacing a Higgs field by a gauge field. This procedure is the counterpart of dimensional reduction.

Of course one can repeat this for tori of other dimensions, keeping in mind that the gauge theory is well defined in low dimensions only.

2.3 N=4 SYM AND M-THEORY ON T^3

The $\mathcal{N} = 4$ SYM model (2.31) has been thoroughly studied in the past. Elegant semi-classical studies have revealed that the spectrum of dyonic BPS-saturated states in the theory exhibits an exact symmetry between electric and magnetic charges [66][69][88][85]

[42]. This S-duality is expected to extend into the full quantum regime, thereby providing an exact mapping between the strong coupling and weak coupling sectors. Although recent breakthroughs in non-perturbative supersymmetric gauge theories and string theory have produced substantial evidence for the duality conjecture, finding an explicit construction of the duality mapping still seems as difficult as ever.

The S-duality of the $\mathcal{N} = 4$ model (2.31) forms an important ingredient in the matrix theory [9] formulation of 11-dimensional M-theory. As reviewed in section 2.2.2, matrix theory proposes a concrete identification between the $U(N)$ SYM model defined on a three-torus T^3 and DLCQ type IIA or IIB string theory compactified on a two-torus T^2. A particularly striking consequence of this conjectured correspondence is that the S-duality of the gauge model gets mapped to a simple T-duality in the string theory language [79][36].

Central in the correspondence with M-theory on T^3 is the following construction of the 11-dimensional supersymmetry algebra in terms of the SYM degrees of freedom. The generators of the four-dimensional $\mathcal{N} = 4$ supersymmetry algebra can be conveniently combined into one single $SO(9,1)$ spinor supercharge Q_α, by considering the 4D SYM model as the dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM theory. In this ten dimensional notation the fermionic fields transform under the supersymmetry according to $\delta\psi = \Gamma^{\mu\nu}F_{\mu\nu}\epsilon$, and the corresponding supercharge Q_α is equal to

$$Q = \int_{T^3} \mathrm{tr}\left[\Gamma^r \psi E_r - \Gamma^0\Gamma^{rs}\psi\frac{1}{2}F_{rs}\right]. \tag{2.39}$$

Here and in the following the indices r, s run from 1 to 9. As the spatial part of our space-time manifold is compact we have an additional global supersymmetry: the action is invariant under adding a constant spinor to ψ via $\delta\psi = \tilde{\epsilon}$. We denote the corresponding supercharge by $\tilde{Q}$

$$\tilde{Q} = \int_{T^3} \mathrm{tr}\,\psi, \tag{2.40}$$

where we put the volume of the three torus (and the length of all its sides) equal to one. The supersymmetry algebra is

$$\begin{aligned} \left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} &= N\delta_{\alpha\beta}, \\ \left\{Q_\alpha, \tilde{Q}_\beta\right\} &= Z_{\alpha\beta}, \\ \left\{\bar{Q}_\alpha, Q_\beta\right\} &= 2\Gamma^0 H + 2\Gamma^i P_i, \end{aligned} \tag{2.41}$$

where the central charge term

$$Z = \Gamma^0\Gamma^i e_i - \Gamma^{ij} m_{ij} \tag{2.42}$$

consists of the total electric and magnetic flux through the three-torus, defined via

$$e_i = \int_{T^3} \mathrm{tr}\, E_i, \qquad m_{ij} = \int_{T^3} \mathrm{tr}\, F_{ij}. \tag{2.43}$$

H in (2.41) denotes the supersymmetric Yang-Mills Hamiltonian and the quantities P_i are the integrated energy momentum fluxes, defined (in 3+1 notation) as

$$P_i = \int_{T^3} \mathrm{tr}(E_j F_{ji} + \Pi^I D_i X_I + \frac{1}{2} i\psi^{\mathrm{T}} D_i \psi), \tag{2.44}$$

where Π_I is the conjugate momentum to the Higgs scalar field X_I. In writing the above supersymmetry algebra we have assumed that the $U(1)$ zero mode part of Π_I vanishes. From the eleven dimensional M-theory perspective, this means that we assume to be in the rest-frame in the uncompactified space directions.

The superalgebra (2.41) has the same form as the one of eleven dimensional supergravity (cf.(2.18)) in the DLCQ formalism:

$$\begin{aligned}
\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} &= p^+ \delta_{\alpha\beta}, \\
\left\{Q_\alpha, \tilde{Q}_\beta\right\} &= 2p^a \gamma_{a\alpha\beta} + 2Z^{a_1 a_2}\gamma_{a_1 a_2 \alpha\beta} + 2Z^{a_1 \cdots a_5}\gamma_{a_1 \cdots a_5 \alpha\beta}, \\
\left\{Q_\alpha, Q_\beta\right\} &= 2p^- \delta_{\alpha\beta} + 2Z^a \gamma_{a\alpha\beta} + 2Z^{a_1 \cdots a_4}\gamma_{a_1 \cdots a_4 \alpha\beta}.
\end{aligned} \tag{2.45}$$

Here we split the 32-component supercharge of 11D supergravity into two sixteen-component supercharges Q and $\tilde{Q}$. As prescribed by DLCQ, x^- is defined on a circle and x^+ plays the role of time. In writing up the supersymmetry algebra we have therefore set the charges with a $+$ component to zero, while the charges with a $-$ component do not necessarily vanish. These charges are indicated by Z^a and $Z^{a_1 \cdots a_4}$ and they belong to membranes respectively five-branes wrapped around the light-like circle and transversal directions. Comparing the superalgebras (2.45) and (2.41) we see that they have the same form, except for the five-brane central charge. But the five-brane is not included in M theory on low dimensional tori, because then it is infinitely massive, so the superalgebras are indeed identical.

We thus get a dictionary for the various fluxes of $3+1$ dimensional SYM theory in terms of charges of eleven-dimensional M-theory on T^3 and 10-dimensional IIA string theory compactified on T^2. This leads to the following list of correspondences (here i, j run from 1 to 2):

$$\begin{array}{|c|c|} \hline N & H \\ \hline e_3 & e_i \\ \hline m_3 & m_i \\ \hline p_3 & p_i \\ \hline \end{array} \quad \leftrightarrow \quad \begin{array}{|c|c|} \hline p_+ & p_- \\ \hline p_9 & p_i \\ \hline m_{ij} & m_{9j} \\ \hline m_{9-} & m_{i-} \\ \hline \end{array} \quad \leftrightarrow \quad \begin{array}{|c|c|} \hline p_+ & p_- \\ \hline q_0 & p_i \\ \hline m_{ij} & w_j \\ \hline w_- & m_{i-} \\ \hline \end{array} \tag{2.46}$$

4D SYM — M-theory on T^3 — IIA string on T^2

Here q_0 denotes the D-particle number in the IIA string theory. In the second and third table, the integers m_{ij} - corresponding to the SYM magnetic fluxes - denote the wrapping numbers of the M-theory membrane around the T^3 and the D2-brane around the T^2 respectively. The M-theory membranes wrapped m_{9j} times around the compact light-like direction turn into IIA strings with winding number w_i.

States with non-zero momentum flux $P_i = p_i$ in the SYM theory correspond to membranes that are wrapped around the longitudinal x^- light-like direction. In the original proposal of [9], M-theory arises in the limit $N \to \infty$ while restricting the spectrum of the $\mathcal{N} = 4$ SYM model to the subspace of states that have energy of order $1/N$. This limit amounts to a decompactification of the longitudinal direction. States with non-zero p_i will thus correspond to infinite energy configurations containing membranes stretched along the light-cone direction. All finite energy states therefore must have $p_i = 0$.

In the DLCQ setup, the longitudinal membranes are no longer infinitely massive, and must be naturally included in the spectrum.

2.3.1 U-Duality in matrix theory on T^2

The correspondence with string theory and M-theory gives a number of predictions concerning the duality properties of the gauge theory. These predictions in particular concern the BPS spectrum of the SYM model at finite N as well the behavior of the model at large N. For example, from the above table 2.46 it is seen immediately that electric-magnetic duality in the SYM theory follows from the T-duality on the two torus, as first noted in [51]. The T-duality that exchanges the D-particles with the D2-branes and the KK momenta with the NS winding numbers in type IIA string theory, when translated to the first table indeed gives rise to the S-duality that interchanges all the electric and magnetic fluxes. Via this correspondence, the complete U-duality symmetry $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ of M-theory on T^3 is expected to be realized as an exact symmetry of the matrix formalism [79][36]. [3] A discussion about the expected $SL(5,\mathbb{Z})$ duality symmetry group of matrix theory on T^4 can be found in [74]. This case is more subtle as the theory is not simply described by $4+1$ dimensional gauge theory, because it is not renormalizable.

At finite N there are reasons to suspect that the duality group that acts on the BPS sector is in fact enlarged. As we have discussed in section 2.2.2, for finite N matrix theory describes M-theory in the DLCQ formalism, and correspondingly one no longer needs to restrict to states with vanishing momentum flux p_i. BPS states at finite N can therefore carry a total of 10 charges, labeled by (N, e_i, m_i, p_i).

The gauge theory descends from Born Infeld theory in the zero slope limit $\alpha' \to 0$ (cf. section 2.1). Therefore we can interpret the $U(N)$ SYM model as a low energy description of all possible bound states of N D3 branes of IIB string theory on T^3. In this correspondence, e_i denotes the NS string and m_i the D-string winding number, while p_i is the KK momentum. The SYM theory should thus encompass all BPS bound states of this system.

It should be emphasized, however, that from this reasoning we should only expect the BPS *degeneracy* formula to be $SL(5,\mathbb{Z})$ symmetric, while of course the energy

[3] For a review of U-duality in matrix theory we refer to [68].

spectrum is not, since the Yang-Mills model only represents a particular limit of the N D3 brane system.

2.4 BPS SPECTRUM

The goal of our study in this section is to determine the explicit form of the BPS degeneracies for finite N as a function of the charges (N, e_i, m_i, p_i), and thereby exhibit its full duality symmetry. We will approach this problem in two ways: first from M-theory and then directly from the $\mathcal{N}=4$ SYM model. Our main finding is that the spectral degeneracy of individual bound states from both points of view is identical, and furthermore exhibits a full $SL(5,\mathbb{Z})$ duality symmetry. Part of this large duality group also acts on the rank N of the gauge group, an example of so called Nahm type transformations [67].

The BPS-states that we will consider respect 1/4 of all supersymmetries. Every such BPS-state in a fixed multiplet satisfies

$$\left(\epsilon^\alpha Q_\alpha + \tilde{\epsilon}^\alpha \tilde{Q}_\alpha\right)|\mathrm{BPS}\rangle = 0, \tag{2.47}$$

for a certain fixed collection of $SO(9,1)$ spinors ϵ and $\tilde{\epsilon}$. These spinors are, up to an overall factor, completely determined by the set of charges.

By taking the commutator with $\bar{Q}$ and $\tilde{Q}$ in the preceding equation we get two conditions for the spinors ϵ and $\tilde{\epsilon}$

$$\begin{aligned} 2\Gamma^0 H\epsilon + 2\Gamma^i P_i \epsilon + \Gamma^0 Z\tilde{\epsilon} &= 0, \\ N\tilde{\epsilon} + Z^\dagger \epsilon &= 0. \end{aligned} \tag{2.48}$$

Plugging $\tilde{\epsilon}$ in the first equation gives the following equation for ϵ

$$\left(\Gamma^0 H' + \Gamma^i P_i'\right)\epsilon = 0, \tag{2.49}$$

where H' and P_i' denote the Hamiltonian and momentum fluxes with the zero-mode contributions removed. Explicitly,

$$\begin{aligned} H &= \frac{1}{2N}(e_i^2 + m_i^2) + H', \\ P_i &= (e \wedge m)_i/N + P_i'. \end{aligned} \tag{2.50}$$

From the last equation, we see that the eigenvalues of P_i' are equal to $p_i' = \kappa_i/N$ with

$$\kappa_i = Np_i - (e \wedge m)_i. \tag{2.51}$$

The operator in equation (2.49) should have eigenvalues equal to zero. This is only the case when the magnitude of H' and the length of P_i' are equal, $H' = |P_i'|$. Using this relation, we can write equation (2.49) in the following way

$$\left(\Gamma_0 + \Gamma^i \hat{\kappa}_i\right)\epsilon = 0, \tag{2.52}$$

where the vector $\hat{\kappa}_i$ denotes the unit vector in the direction of κ_i.

2.4.1 BPS SPECTRUM FROM M-THEORY

Now we are ready to determine the detailed BPS spectrum from discrete light-cone M-theory. To this end it is useful to introduce the notion of an *irreducible* BPS state, as a state that contains only one *single* BPS bound state. Indeed, general BPS states can in principle combine more than one such bound state into one second quantized BPS state. The combined state will still be BPS, provided each of the irreducible constituent bound states is left invariant by the *same* set of supersymmetries, *i.e.* with the same spinors ϵ and $\tilde{\epsilon}$.

The degeneracy of irreducible BPS states is determined almost uniquely from U-duality invariance, and from the known BPS spectrum of perturbative string states. The relevant U-duality group for our case turns out to be as large as the complete $SL(5,\mathbb{Z})$ duality group of string theory on T^3, *i.e.* M-theory on T^4.

The 10 quantum numbers (N, e_i, m_i, p_i) can be combined into a 5×5 anti-symmetric matrix [92]

$$\begin{pmatrix} 0 & p_3 & -p_2 & e_1 & m_1 \\ -p_3 & 0 & p_1 & e_2 & m_2 \\ p_2 & -p_1 & 0 & e_3 & m_3 \\ -e_1 & -e_2 & -e_3 & 0 & N \\ -m_1 & -m_2 & -m_3 & -N & 0 \end{pmatrix}, \tag{2.53}$$

on which $SL(5,\mathbb{Z})$ acts by simultaneous left- and right-multiplication. [4] The smaller $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$ symmetry group is included in the group $SL(5,\mathbb{Z})$ and is represented by block-diagonal matrices, with an $SL(3,\mathbb{Z})$ group element in the upper left corner and an $SL(2,\mathbb{Z})$ element in the lower right corner. Together with an $SL(5,\mathbb{Z})$ matrix that is not in this block-diagonal form the elements of $SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$ generate the complete extended duality group. We will call the extra transformations Nahm type transformations, because they mix rank and electro-magnetic fluxes with each other. They resemble the well known Nahm duality mapping that exchanges instanton number and rank of gauge group in Euclidean $U(N)$ Yang-Mills theory [19]. We come back to this issue in section 2.5.

The following bilinear combinations of the fluxes

$$K_i = (Np_i - (e\wedge m)_i,\ p\cdot m\ ,\ p\cdot e\ ), \tag{2.54}$$

transform as a 5 vector under $SL(5,\mathbb{Z})$ duality. It can be formed out of the fluxes by contracting the matrix (2.53) with its Hodge dual, a three tensor. From this we deduce that the relevant $SL(5,\mathbb{Z})$ invariant scalar combination we can make out of the ten charges is the integral length of this five-vector [5]

$$|K| = \gcd(Np_i - (e\wedge m)_i,\ p\cdot m\ ,\ p\cdot e\ ). \tag{2.55}$$

[4] The $SL(5,\mathbb{Z})$ transformations also acts on the metric of the torus and the coupling constant in a non-trivial way. We will discuss this in the last section of this chapter.

[5] Another $SL(5,\mathbb{Z})$ scalar is $\gcd(N, p_i, e_i, m_i)$ which is not relevant for the degeneracies of BPS states in Yang-Mills theory. This scalar will be important however when one considers the BPS spectrum of supersymmetric Born Infeld theory on T^3 [92].

Duality invariance thus predicts that the degeneracy of irreducible BPS bound states should be expressible in terms of the quantity $|K|$ only.

To determine the explicit degeneracy formula, we note that by using the U-duality symmetry, any irreducible bound state can be rotated into a state that carries only KK momentum and NS string winding (in the SYM language these states have zero momenta $p_i = 0$). Such a state must necessarily be made up from a single fundamental IIA string. The invariant $|K|$ for this perturbative string state simply reduces to the bilinear combination of momenta and winding numbers that determines (via the BPS restriction) the oscillator level of the string. Single BPS states of toroidal compactified II string theory have only left-moving (or right-moving) excitations. The number of irreducible BPS bound states is therefore counted by means of the chiral string partition function [72]

$$\sum_K c(K) q^K = (16)^2 \prod_n \left(\frac{1+q^n}{1-q^n}\right)^8 , \tag{2.56}$$

with $K = |K|$ as defined in (2.55) .

General BPS states of discrete light-cone gauge M-theory may consist of more than one irreducible bound state. The total BPS condition requires that the charges of these separate bound states must be compatible. The complete second quantized partition sum is obtained by taking into account all possible such ways of combining individual bound states into a second quantized configuration with a given total charge. More detailed comments on the combinatorial structure of the second quantized BPS partition sum will be given in section 2.4.3.

2.4.2 BPS spectrum from N=4 SYM on T^3

The above description of the BPS spectrum can be reproduced directly from the $U(N)$ gauge theory on T^3 as follows. First let us recall the definition of the electro-magnetic flux quantum numbers. To this end it is useful to decompose the $U(N)$ gauge field into a trace and a traceless part

$$A_\mu = A_\mu^{U(1)} \mathbb{1} + A_\mu^{SU(N)} \tag{2.57}$$

and to allow the fields on T^3 to be periodic up to gauge transformations of the form

$$\begin{aligned} A_i^{SU(N)}(x + a_j) &= \Omega_j A_i^{SU(N)}(x) \Omega_j^{-1}, \\ A_i^{U(1)}(x + a_j) &= A_i^{U(1)}(x) - 2\pi m_{ij}/N, \end{aligned} \tag{2.58}$$

with m_{ij} integer. Here a_i with $i = 1, 2, 3$ denote the translation vectors that define the three torus T^3. [6] The $SU(N)$ rotations Ω_i must satisfy the $\mathbb{Z}_N$ cocycle conditions

$$\Omega_j \Omega_k = \Omega_k \Omega_j e^{2\pi i \mu_{jk}/N}, \tag{2.59}$$

[6] We repeat here that for simplicity we mostly take T^3 to be cubic with sides of length 1.

for integer μ_{jk}. The quantities μ_{ij}/N define the 't Hooft $\mathbb{Z}_N$ magnetic fluxes [58] and m_{ij}/N can be identified with the $U(1)$ magnetic flux defined in (2.43). The integers μ_{ij} and m_{ij} are restricted via the condition that the combined gauge transformation (2.58) defines a proper $U(N)$ rotation. This requirement translates into the Dirac quantization condition that the total flux $(\mu_{ij} - m_{ij})/N$ must be an integer. Hence $m_{ij} = \mu_{ij} (\text{mod } N)$.

Electric flux carried by a given state is defined via the action of quasi-periodic gauge rotations $\Omega[\mathbf{n}]$ defined via

$$\Omega_j \Omega[\mathbf{n}] = \Omega[\mathbf{n}] \Omega_j e^{2\pi i n_j/N}, \tag{2.60}$$

with integer n_j. Such gauge rotations will preserve the boundary condition (2.58) on the gauge fields. A state $|\psi\rangle$ is defined to carry $SU(N)$ flux ϵ_j if it satisfies the eigenvalue condition

$$\widehat{\Omega}[\mathbf{n}] |\psi\rangle = e^{\frac{2\pi i}{N} n^j \epsilon_j} |\psi\rangle, \tag{2.61}$$

where $\widehat{\Omega}[\mathbf{n}]$ denotes the quantum operator that implements the gauge rotation $\Omega[\mathbf{n}]$ on the state $|\psi\rangle$. Similarly as for the magnetic flux, the electric flux receives an overall $U(1)$ contribution e_i defined in (2.43). The abelian and non-abelian parts of the flux must again be related via $e_i = \epsilon_i (\text{mod } N)$.

To determine the supersymmetric spectrum for a given set of charges, the idea is to first reduce the phase space of the $U(N)$ SYM model to the space of classical supersymmetric configurations and then to quantize this BPS reduced phase space. The justification for this procedure should come from the high degree of supersymmetry in the problem, while furthermore the degeneracy of BPS states is known to be a very robust quantity.

To obtain the reduced phase space, we recall that the SUSY transformation for the fermionic partners of the Yang-Mills fields reads (again using ten-dimensional notation)

$$\delta\psi = \left(E_r \Gamma^{0r} + \frac{1}{2} F_{rs} \Gamma^{rs} \right) \epsilon + \tilde{\epsilon}. \tag{2.62}$$

The BPS restriction requires that the right-hand side vanishes for those ϵ and $\tilde{\epsilon}$ determined in the previous section. Thus in particular we can use the equation (2.48) to express $\tilde{\epsilon}$ in terms of ϵ and the $U(1)$ zero modes. The result is

$$\delta\psi = \left(E'_r \Gamma^{0r} + \frac{1}{2} F'_{rs} \Gamma^{rs} \right) \epsilon = 0. \tag{2.63}$$

where the primed quantities are equal to the un-primed ones with the constant $U(1)$ parts removed. For BPS-states in a fixed multiplet, supersymmetry is unbroken for ϵ satisfying the equation (2.49) above. Hence for these BPS-states the following must hold for all spinors

$$\left(E'_r \Gamma^{0r} + \frac{1}{2} F'_{rs} \Gamma^{rs} \right) \left(\Gamma^0 - \Gamma^k \hat{\kappa}_k \right) \epsilon = 0. \tag{2.64}$$

Note that $\Gamma^0 - \Gamma^k \hat{\kappa}_k$ acts like a projection operator on the space of spinors satisfying equation (2.49). We conclude that the matrix in spinor space in the last equation has to vanish. This is the case when E' and F' satisfy the following two conditions

$$\begin{aligned} E'_i \hat{\kappa}^i &= 0, \\ E'_{[r} \hat{\kappa}_{s]} &= F'_{rs}. \end{aligned} \tag{2.65}$$

From now one we shall omit the prime, and simply denote by E and F_{ij} the $U(N)$ fields without the $U(1)$ constant mode. We will also return to a 3+1-dimensional notation, and for additional notational convenience, use the $SL(3,\mathbb{Z})$ symmetry to rotate the three-vector κ_i defined in (2.51) in the 3 direction. So we will choose coordinates such that

$$\begin{aligned} \kappa_3 &= Np_3 - e_i m_{i3}, \\ \kappa_j &= Np_j - e_i m_{ij} - e_3 m_{3j} = 0 \qquad i,j = 1,2. \end{aligned} \tag{2.66}$$

Here and from now on the indices i,j run from 1 to 2. From the second condition in (2.65) we read off that the gauge and Higgs fields are flat on the plane perpendicular to $\hat{\kappa}$, meaning

$$\begin{aligned} F_{ij} &= 0, \\ D_i X_J &= 0, \\ [X_I, X_J] &= 0. \end{aligned} \tag{2.67}$$

In addition we have

$$\begin{aligned} E_i &= F_{3i}, \\ \Pi_I &= D_3 X_I. \end{aligned} \tag{2.68}$$

Finally, the first condition in (2.65) simply becomes

$$E_3 = 0. \tag{2.69}$$

We can interpret this constraint equation as a gauge invariance condition under arbitrary local shifts in the longitudinal gauge field A_3. We can therefore exploit this gauge invariance by putting $A_3 = 0$. The relations (2.67) then simplify to the statement that the transversal gauge fields A_i and Higgs scalars X_I satisfy the chiral 2D free field equations

$$\begin{aligned} \partial_0 A_i &= \partial_3 A_i, \\ \partial_0 X_I &= \partial_3 X_I. \end{aligned} \tag{2.70}$$

Furthermore, the equations of motion imply that the fields do not depend on the coordinates of the transverse torus. Thus we conclude that the BPS reduced theory is described by the left-moving chiral sector of a two-dimensional sigma model with target space given by the space of solutions to the flatness conditions (2.67) subject to the twisted boundary conditions specified by the electro-magnetic flux quantum numbers.

To determine the detailed properties of this sigma model, let us first consider the case with all electro-magnetic fluxes equal to zero. The only non-zero quantum numbers are therefore N and p_3. In this case we can parameterize the space of solutions to (2.67) by means of the orbifold sigma model on the N-fold symmetric product space

$$\frac{(\mathbb{R}^6 \times T^2)^N}{S_N}, \tag{2.71}$$

where S_N denotes the permutation group of N elements, acting on the N copies of the transversal space $T^2 \times \mathbb{R}^6$. [7] To see that this is the right space, we observe that for solutions to (2.67) one can always choose a gauge in which all $U(N)$ valued fields take the form of diagonal matrices. Each such matrix field thus combines N separate scalar fields, corresponding to the N eigenvalues. The S_N permutation symmetry arises as a remnant of $U(N)$ gauge invariance, acting via its Weyl subgroup on the space of diagonal matrices. Finally, the flat transversal gauge fields A_i with $i = 1, 2$ give rise to *periodic* 2D scalar fields, since constant shifts in A_i by multiples of 2π are pure gauge rotations.

The model thus reduces to the free limit of type IIA matrix string theory [65][10][29] in the discrete light-cone gauge [80]. As explained in section 1.2 the Hilbert space of the model decomposes into twisted sectors labeled by the partitions of N, in which the eigenvalue fields combine into a collection of 'long strings' of individual length n_k such that the total length adds up to $\sum_k n_k = N$. Each such string is made up from, say, n_k eigenvalues that, by their periodicity condition around the 3-direction are connected via a cyclic permutation of order n_k. In the M-theory interpretation, all these separate strings will indeed correspond to separate bound states, *i.e.* particles that each can move independently in the uncompactified space directions. The general form of the BPS partition function of symmetric product sigma models of the form (2.71) has been described in detail in [28].

In the following we will mainly concentrate on the irreducible states, describing one single BPS particle. These necessarily consist of one single string of maximal length. In the present case, with zero total electro-magnetic flux, this maximal string has total winding number N around the 3-direction. Correspondingly, its oscillation modes have energies that are quantized in units of $1/N$. Thus the degeneracy of states as a function of N and p_3 is obtained by evaluating the chiral superstring partition function, as given in (2.56), at oscillator level Np_3.

Next let us turn on the magnetic flux m_3. The space of solutions to (2.67) with twisted boundary conditions (2.58) around the transverse torus again takes the form of a symmetric product

$$\frac{(\mathbb{R}^6 \times T^2)^{N'}}{S_{N'}}, \tag{2.72}$$

but where now $N' = \gcd(N, m_3)$. In order to visualize this reduction, we note that gauge rotations Ω_i with $\Omega_1\Omega_2 = \Omega_2\Omega_1 \exp(2\pi i m_3/N)$ that define the twisted boundary conditions, can be chosen to lie within an $SU(k)$ subfactor of $U(N)$ where $k =$

[7] This orbifold sigma model was first considered in relation with $\mathcal{N} = 4$ SYM theory on the three torus in [51].

$N/\gcd(N, m_3)$. By decomposing the matrix valued fields according to the action of $SU(k) \otimes U(N')$ with $N' = \gcd(N, m_3)$, we can thus factor out a sector of $U(N')$ valued field variables that are unaffected by the twisted boundary conditions. Now following the same reasoning as before, these fields parameterize the symmetric product space of the above form. Note that in the particular case that $m_3 = 1$, the whole $SU(N)$ part of the moduli space of solution to (2.67) collapses to a point, so that only the $U(1)$ part survives.

By a very similar reasoning we can also include the electric flux e_3 in our description. Like with the magnetic flux, an electric flux $e_3 = 1$ has the effect of reducing the $SU(N)$ part of the vacuum moduli space (2.67) to a point, or rather, it projects out just one single supersymmetric state in the $SU(N)$ sector [87]. More generally, however, it can be seen that one can again factor out a $U(N')$ subfactor of the model, that is unaffected by both the electric and magnetic flux e_3 and m_3, where now

$$N' = \gcd(N, m_3, e_3). \tag{2.73}$$

The BPS sector for non-zero m_3 and e_3 is thus obtained by quantizing the supersymmetric orbifold sigma model on (2.72), with N' equal to (2.73).

The spectrum of irreducible BPS bound states is obtained as before, by considering the Hilbert space sector defined by the eigenvalue string of maximal length. The maximal winding number is now equal to N'. The total momentum along this string is determined by the remaining quantum numbers of the BPS state, and should be equal to

$$p_3 - e_i m_{i3}/N, \tag{2.74}$$

which is the total momentum of the BPS state minus the contribution from the $U(1)$ electro-magnetic fluxes. Notice that the latter contribution is in general fractional. However, since the oscillation modes of the long string states also have fractional oscillation number quantized in units of $1/N'$, this fractional total momentum (2.74) can in fact be obtained via integer string oscillation levels. The total oscillation level corresponding to this momentum flux is

$$K = N' \times (p_3 - e_i m_{i3}/N) \tag{2.75}$$

and it is easy to check using (2.66) that this is an integer. In fact, after taking the direction of κ_i again arbitrary, we find that the integer quantity (2.75) becomes equal to the $SL(5,\mathbb{Z})$ invariant length $|K|$ defined in (2.55) of the five-vector (2.54). In this way we reproduce the description of the BPS spectrum given in the previous section. In particular we find confirmation that the degeneracy of supersymmetric bound states has a discrete $SL(5,\mathbb{Z})$ duality invariance.

2.4.3 Complete BPS partition function

As mentioned earlier, the complete supersymmetric spectrum of the $\mathcal{N} = 4$ SYM model contains many more sectors, that in M-theory describe configurations of multiple BPS bound states. These correspond to the other twisted sectors in the orbifold model

(2.72), describing multiple strings with separate lengths n_k with $\sum_k n_k = N'$. Via their zero modes these strings each separately carry all the possible flux quantum numbers. The degeneracy of these separate states as a function of these charges is identical to the $SL(5,\mathbb{Z})$ invariant result just described. However, it turns out that the combinatorics by which many such states can be combined into one second quantized BPS state no longer respects the full $SL(5,\mathbb{Z})$ symmetry. For this it would be necessary that, via the compatibility of the individual BPS conditions, the 10 dimensional charge vectors of all constituent states must align in the same direction. It can be seen, however, from the second condition in (2.48) and condition (2.49) that this alignment is not entirely implied: in the above notation, all charges must indeed align, *except* for the individual p_3 momenta.

In case of nonzero p_3 momentum (and other fluxes equal to zero) the multiple states are in one-to-one relation with partitions of rank N. One can distribute the p_3 momentum freely among the different sectors. We conclude that the degeneracies are simply determined by a second quantized string partition function

$$\sum_{N,p_3} p^N q^{p_3} = \prod_{r,s} \frac{1}{(1-p^r q^s)^{c(rs)}}, \tag{2.76}$$

where the integers $c(K)$ are the degeneracies of long string states defined by equation (2.56).

In the situation where we have general fluxes, the multiple BPS states are, as explained above due to supersymmetry, related to partitions $\sum_k n'_k = n'$, with n' defined by the greatest common divisor of all charges except p_3

$$n' := \gcd(N, e_i, m_i, p_1, p_2). \tag{2.77}$$

A single short string has length $n_k = n'_k N'/n'$ and oscillator level $\tilde{p}^3_k = p^3_k - \frac{n'_k}{n'}\frac{(e\times m)_3}{N}$. Its degeneracy is equal to $d(n_k \tilde{p}^3_k)$. Again the way the p_3 momentum is divided among the different short string sectors is not limited by supersymmetry.

Hence the total BPS partition function is again generated by a second quantized string partition function

$$\sum_{n',p_3} p^{n'} q^{p_3} = \prod_{r,s} \frac{1}{(1-p^r q^s)^{d(s_r)}}, \tag{2.78}$$

where

$$s_r = r\frac{N'}{n'}\left(s - \frac{r}{n'}\frac{(e\times m)_3}{N}\right). \tag{2.79}$$

The degeneracy formula (2.78) is invariant under the Nahm type transformation that exchanges m_3 and N. We therefore conclude that the BPS partition function of the $U(N)$ SYM model does not exhibit the $SL(5,\mathbb{Z})$ symmetry, but still has a symmetry essentially larger than the manifest $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$. In the next section we will discuss the geometric origin of this extra symmetry in some detail.

2.5 Nahm duality

An especially interesting class of duality transformations that act on BPS sectors of $U(N)$ SYM theory are those that interchange the rank N of the gauge group with the magnetic flux.

In the notation used in section 2.4.2 (with $\hat{\kappa}$ rotated in the 3-direction) the BPS reduced quantum phase space exhibits a manifest symmetry under the interchange

$$N \to m_3 \qquad m_3 \to -N \qquad m_i \to \epsilon^{ij} m_j \qquad e_i \leftrightarrow p_i, \tag{2.80}$$

as well as under its electro-magnetic dual counterpart

$$N \to e_3 \qquad e_3 \to -N \qquad e_i \to \epsilon^{ij} e_j \qquad m_i \leftrightarrow p_i. \tag{2.81}$$

In particular one can verify that the last two relations and κ_3/N in (2.66) as well as the integers N' in (2.73) and n defined in (2.77) are invariant under these two mappings.

The second type of duality symmetry (2.81) is particularly interesting, because from the M-theory perspective, this symmetry (2.81) must be some manifestation of 11-dimensional covariance, as it exchanges the eleventh direction (more precisely a light direction) with a transversal direction.

Because of the possible relevance for Lorentz invariance of matrix theory, it is of interest to know whether the symmetry extends to the full $\mathcal{N} = 4$ model. Although we do not expect that this is the case, [8] we cannot resist reviewing the geometrical origin of this duality mapping, which can in fact be defined for arbitrary non-BPS gauge configurations.

The type of transformations (2.80) and (2.81) are very similar to the Nahm-type transformations, considered e.g. in [19]. There it was proven that a k-instanton solution of $U(n)$ Euclidean gauge theory on a four torus can be mapped to an n-instanton solution of $U(k)$ gauge theory on the dual four torus. This mapping was shown to induce a bijection between the moduli spaces of the two instantons. In two dimensions it is possible to formulate an analogous mapping, where now rank and magnetic flux of an arbitrary gauge field configuration are exchanged.

Consider an arbitrary $U(N)$ gauge field $A = A_x + iA_y$ on T^2 with magnetic flux M. This configuration can be related to a dual $U(M)$ gauge field with flux N defined on the dual torus $\hat{T}^2$ as follows. The key idea of the construction is to consider the parameter family of connections on T^2 of the form (here a, b denote the $U(N)$ color indices, we use the fundamental representation of the gauge group $U(N)$)

$$A^{ab}(x; z) = A^{ab}(x) + 2\pi \mathbf{1}^{ab} z \tag{2.82}$$

with $z = z_1 + iz_2$ a complex coordinate on the dual torus $\widehat{T}^2$. The mapping proceeds by considering the space of zero modes of the Dirac-Weyl equation defined by A. Let D and D^* denote the corresponding Dirac operator acting on right- and left-moving

[8] In the context of gauge theories on non-commutative tori, Nahm-type transformations are part of a larger manifest (classical) duality symmetry [54][55]. We will briefly comment on this at the end of this section.

spinors, respectively. We will assume that D^* has no zero-modes (we can make this assumption because, depending on the sign of the magnetic flux, either DD^* or D^*D is a strictly negative definite operator on the two torus). According to a Dirac index theorem [47], the number of zero-modes of D then equals the magnetic flux M. We have

$$D\psi_i(x;z) = 0, \tag{2.83}$$

with $i = 1, \dots, M$. We can choose an orthonormal basis of zero modes, satisfying the orthogonality relations

$$\int d^2x\, \overline{\psi}_i(x;z)\psi_j(x;z) = \delta_{ij}. \tag{2.84}$$

The dual $U(M)$ gauge field $\widehat{A}$ on $\widehat{T}^2$ is now defined as

$$\widehat{A}_{ij}(z) = \int d^2x\, \overline{\psi}_i^a(x;z)\widehat{\partial}\psi_{aj}(x;z), \tag{2.85}$$

with $\widehat{\partial} = \frac{\partial}{\partial z}$. This definition is equivalent to the following formula for the dual covariant derivative $\widehat{D}$

$$\widehat{D}\psi = (\mathbf{1} - \mathbf{P})\widehat{\partial}\psi, \tag{2.86}$$

where $\mathbf{P}$ denotes the projection on the space of zero modes ψ_i of the operator D. It can be shown that this dual gauge field has magnetic flux equal to N, and moreover that the mapping from A to $\widehat{A}$ is a true duality (it squares to the identity).

To make these properties more manifest, it is useful to obtain a somewhat more explicit form of the dual covariant derivative $\widehat{D}$. To this end, define the Green function

$$\Delta G(x,y) = \delta^{(2)}(x-y) \tag{2.87}$$

of the Laplacian $\Delta = DD^*$, and introduce the notation

$$(\mathbf{G}\psi)(x;z) = \int d^2y\, G(x,y)\psi(y;z). \tag{2.88}$$

Then the projection operator $\mathbf{P}$ satisfies the relation

$$\mathbf{1} - \mathbf{P} = D^*\mathbf{G}D. \tag{2.89}$$

Inserting this identity into the analogous formula for $\widehat{D}^*\psi$ of the definition (2.86), we find that

$$\widehat{D}^*\psi = D^*\mathbf{G}D\hat{\partial}^*\psi = 0, \tag{2.90}$$

since $[D, \hat{\partial}^*] = 0$. Hence the zero modes of the Dirac-Weyl operator on the dual torus are equal to the ones on the original torus, with opposite chirality. By an explicit construction [53] one can indeed verify that the dual $U(M)$ gauge field has magnetic flux equal to N, as predicted by the index theorem. Thus the Nahm transformation can be summarized in an elegant way by means of the two equations (2.83) and (2.90), which together specify the map from A to $\widehat{A}$. Note in particular that in this form, one of the 'magical' properties of the Nahm transformation has become manifest, namely that it is a map of order 2, *i.e.* it squares to the identity:

$$\widehat{\widehat{A}} = A, \tag{2.91}$$

up to gauge equivalence. Notice further that the mapping is defined for arbitrary connections A.

When we translate the above construction back to our 3+1-dimensional setting, the resulting mapping indeed interchanges N and m_3 as advocated. In addition, since it is a mapping from T^2 to the dual torus $\widehat{T}^2$, the definition of the momentum flux in the direction of the T^2 gets interchanged with that of the electric flux along this direction, as indicated in (2.81). Naturally, the electric flux represents the conjugate momentum to the constant mode of the gauge fields A_i, and thus defines the total momentum on the dual torus $\widehat{T}^2$. Further inspection also shows that the magnetic fluxes m_i get reflected, as predicted.

Finally, we should of course note that in the interpretation of the $U(N)$ SYM model as describing N D3 branes, this Nahm duality is nothing other than a double T-duality along the 1-2 directions of the three torus. [9] From this perspective, it seems somewhat surprising that the Yang-Mills theory (via the Nahm transformation) still knows about this T-duality symmetry, despite arising from string theory via the zero-slope limit. We will give an explanation for this in the next section.

2.5.1 Nahm duality for gauge theory on noncommutative tori

We end this section with some remarks on Nahm-type duality transformations of gauge bundles defined on non-commutative tori. These gauge theories are relevant for compactifications of matrix theory with non-zero B fields [20][33]. It is therefore of interest to investigate their duality symmetries.

As we have seen above the Nahm transformation on commutative tori is somewhat involved. It requires knowledge about the zero modes of the Dirac equation. On non-commutative tori however, Nahm type transformations can be made manifest [54][55]. These transformations are part of a larger $SO(d, d)$ symmetry, that is known in the mathematics literature as Morita equivalence.

The most simple example of a non-commutative space is the non-commutative two torus $\mathbb{T}^2_\theta$. The coordinates of this torus do not commute, but they satisfy the commutation relation

$$[x^1, x^2] = \frac{\theta}{2\pi i} \quad \text{or} \quad U_1 U_2 = e^{2\pi i\theta} U_2 U_1. \tag{2.92}$$

[9] To recognize this interpretation of the mapping (2.80), see the translation code summarized on page 57. This interpretation of the Nahm transformation, as related to T-duality in string theory, was first suggested in [35].

Here θ is a real parameter and $U_i = e^{2\pi i x^i}$ are exponentials that generate the Fourier modes of the functions which can be defined on the non-commutative torus.

Trivial gauge bundles can be constructed by defining the gauge connection by $\nabla_j = \partial_j + iA_j(\tilde{x})$ where $\tilde{x}^i = x^i + \frac{i\theta}{2\pi}\epsilon^{ij}\partial_j$ are modified coordinates that commute with the original coordinates. The connection depends on the modified coordinates and not on the coordinates x^i, because this guarantees that the connection commutes with the full algebra of functions and therefore obeys the correct Leibniz rule [54][55].

To construct a non-trivial gauge bundle with magnetic flux M modified Fourier modes have to be used to construct a gauge field that has the right properties (for example the translation operators should act like gauge transformation on the fields, as in (2.58)). Now the interesting thing is that these modified Fourier modes themselves generate an algebra of an abelian gauge theory on a non-commutative torus, with modified $\hat{\theta}$-parameter that is related to the original θ via an fractional $SL(2,\mathbb{Z})$ transformation [54][55]. The gauge fields and fluxes also transform in a representation of $SL(2,\mathbb{Z})$. We can turn this around, by starting with an abelian theory defined on a particular non-commutative torus, apply an arbitrary $SL(2,\mathbb{Z})$ transformation on the gauge fields and the non-commutative parameter, and thus get a $SL(2,\mathbb{Z})$ family of equivalent gauge bundles. A Nahm transformation interchanging rank and magnetic flux of the gauge bundle is included in this $SL(2,\mathbb{Z})$ group. Thus Nahm duality can be made manifest in gauge theories on non-commutative spaces.

Analogous results hold for gauge bundles on 4 dimensional non-commutative tori.

The degeneracies of 1/4 BPS states of $U(N)$ gauge theory on a non-commutative three-torus have been calculated in [62]. The result is the same as (2.56) in this chapter. This confirms the idea that the degeneracies of BPS states are rather robust quantities.

2.6 Relation to Born Infeld theory

In this chapter we have shown that the multiplicities of single BPS-states of $\mathcal{N}=4$ supersymmetric Yang Mills theory on T^3 has the extended $SL(5,\mathbb{Z})$ U-duality invariance. This result may be remarkable, but it is clear that a possible explanation comes from the relation between Yang-Mills theory and Born-Infeld theory. Yang-Mills theory is a truncated theory of D-branes, whereas Born-Infeld is a more accurate low energy effective description.

This is most clearly reflected in the BPS mass spectrum of the two theories: the BPS mass spectrum of BI theory compactified on a three torus has the expected $SL(5,\mathbb{Z})$ invariance, as has been shown in [56][92]. This duality symmetry is broken however in the limit $\alpha' \to 0$, that is needed to get SYM theory. Yet the symmetry in the degeneracies of irreducible BPS states remains preserved in the zero slope limit. It is the aim of this last section to explain how this can happen.

2.6.1 Born Infeld BPS mass spectrum

Let us first summarize the analysis that was done in [56] and [92] where the BI-theory BPS mass spectrum was calculated explicitly. The spectrum was derived by starting from the bosonic action with abelian gauge group, (cf. (2.8))

$$S_{BI} = T_3 \int d^3x dt\ e^{-\phi} \sqrt{\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}. \qquad (2.93)$$

For given electro-magnetic fluxes and momenta, the energies of the BPS states can be obtained from the BI Hamiltonian by a Bogomolny type of argument [56]. These masses are in agreement with the BPS masses that were obtained from the NS five-brane in [26][27]. In units where $2\pi\alpha' = 1$ the BPS mass spectrum is [56][92]

$$\begin{aligned} E^2_{\rm BPS} = {} & g_s^{4/5}(\det G)^{-1/5}\left(\frac{1}{g_s^2}(N^2 \det G + G_{ij}m^i m^j) + e^i G_{ij} e^j + p_i G^{ij} p_j\right) + \\ & 2g_s^{-1/5}(\det G)^{-1/5}\sqrt{\det G\ \kappa_i G^{ij}\kappa_j + (p\cdot m)^2 + g_s^2(p\cdot e)^2}. \end{aligned} \qquad (2.94)$$

Here G_{ij} is the string metric. Note that we have put the rank N in the mass formula by hand. The mass spectrum (2.94) is invariant under the complete $SL(5,\mathbb{Z})$ symmetry. Part of this U-duality symmetry is the electric-magnetic S-duality under which the coupling constant is inverted and the string metric gets transformed [16] in the following way

$$g_s \to g_s^{-1}, \qquad G_{ij} \to \frac{1}{g_s} G_{ij}. \qquad (2.95)$$

The other non-trivial symmetry of the mass spectrum (2.94) is a double T-duality (e.g. in the 1 and 2 direction), that as explained in section 2.5, can be interpreted in gauge theory, as a Nahm transformation. Under this duality the string metric and the string coupling constant transform according to [16]

$$G_{11} \to G_{11}^{-1}, \qquad G_{22} \to G_{22}^{-1}, \qquad g_s \to g_s(G_{11}G_{22})^{-1/2}. \qquad (2.96)$$

When combined with the transformations of the fluxes it can be easily checked that the Born Infeld BPS mass spectrum is invariant under S-duality and Nahm transformations. Together with the manifest $SL(3,\mathbb{Z})$ symmetry of the torus these duality transformations generate the complete $SL(5,\mathbb{Z})$ symmetry, and therefore the mass spectrum (2.94) is indeed U-duality invariant. Analogous duality symmetries hold for Born Infeld theory on 4 dimensional and 2 dimensional tori. For example in four dimensions the symmetry group of the BPS mass spectrum is $SO(5,5,\mathbb{Z})$, the U-duality group of string theory in 6 dimensions [56].

The invariance can be made manifest by writing the BPS masses in the following way [92]

$$E^2_{\rm BPS} = -\frac{1}{2}{\rm tr}(\mathcal{G}\mathcal{F}\mathcal{G}\mathcal{F}) + 2\sqrt{K_\rho \mathcal{G}^{\rho\sigma} K_\sigma}. \qquad (2.97)$$

Here $\mathcal{F}$ is the 5×5 matrix (2.53) containing the fluxes, $K_\sigma = (Np_i - (e\wedge m)_i, p\cdot m, p\cdot e)$ is the five vector (2.54) and $\mathcal{G}$ is defined by

$$\mathcal{G} = \begin{pmatrix} (\det g)^{-1/2} g_{ij} & & \\ & g_s(\det g)^{1/4} & \\ & & g_s^{-1}(\det g)^{1/4} \end{pmatrix}, \qquad (2.98)$$

The metric g_{ij} is related to the string metric G_{ij} via

$$G_{ij} = g_s^{1/2} (\det g)^{-1/8}\ g_{ij}. \tag{2.99}$$

On both matrices $\mathcal{F}$ and $\mathcal{G}$ the U-duality group $SL(5,\mathbb{Z})$ acts by conjugation, and hence the $SL(5,\mathbb{Z})$ invariance becomes manifest.

It is instructive to extract the SYM BPS mass spectrum from Born Infeld theory. To this end we reintroduce the string scale α' and expand the expression (2.94)

$$\begin{aligned} E_{\mathrm{BPS}} = {} & (2\pi\alpha')^{-5/4} N V^{1/2} + \frac{1}{2NV}\left(\frac{1}{g_s} m^i g_{ij} m^j + g_s e^i g_{ij} e^j + 2\pi\alpha'^{5/4} V^{1/2} p_i g^{ij} p_j\right) + \\ & \frac{1}{N}\left(\kappa_i g^{ij} \kappa_j + 2\pi\alpha'^{5/4} g_s^{-1} V^{-3/2} (p\cdot m)^2 + 2\pi\alpha'^{5/4} g_s V^{-3/2} (p\cdot e)^2\right)^{1/2} + \cdots, \end{aligned} \tag{2.100}$$

where the volume is defined in terms of the metric, $V = \sqrt{\det g}$. This expansion shows that in the large N limit (or large volume limit) and the string decoupling limit $\alpha' \to 0$ we recover the BPS mass spectrum of SYM,

$$E_{\mathrm{BPS}} = \frac{1}{2NV}\left(g_{\mathrm{YM}}^2 e^i g_{ij} e^j + \frac{1}{g_{\mathrm{YM}}^2} m^i g_{ij} m^j\right) + \sqrt{p'_i g_{ij} p'_j}, \tag{2.101}$$

where $p'_i = p_i - (e\times m)_i/N = \kappa_i/N$. We identified the string coupling constant g_s with the Yang-Mills coupling constant squared

$$g_{YM}^2 = g_s. \tag{2.102}$$

Obviously the SYM mass spectrum (2.101) is invariant under $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ but not under the complete $SL(5,\mathbb{Z})$ U-duality group. For example the Nahm type transformation that exchanges rank N and electric flux e_3 does not leave the mass spectrum invariant. From the matrix theory point of view this is no surprise (at least for finite N) as for that theory the flux N is interpreted as a light-cone momentum, whereas e_3 is momentum in a spatial direction. M(atrix) theory is not likely to be invariant under the exchange of a light-like direction and a space-like direction, so we do not expect the BPS mass spectrum to be invariant under Nahm-type transformations.

2.6.2 Degeneracies

What about the degeneracies of the 1/4 BPS states in Born-Infeld theory? For this theory the same philosophy that was used in the SYM case can be applied: first the phase space of the model is reduced to the space of classical supersymmetric configurations, and then this BPS reduced phase space is quantized.

When doing this analysis an immediate important difference emerges between the BI case and the YM case: As emphasized in section 2.2.2 the superalgebra of SYM is equal to the one of 11D supergravity in the DLCQ formalism, whereas the superalgebra of Born Infeld theory [10] is just the 11D superalgebra with 4 spatial directions compactified [92]. The last mentioned superalgebra has an $SL(5,\mathbb{Z})$-invariance and therefore

[10] We repeat here that the action for supersymmetric abelian BI theory has been constructed [1]. We make the assumption that results for the non-abelian case follow from the analysis of the abelian model.

we expect that the degeneracies of the BI states exhibit this duality symmetry as well. Another difference is of course the square root in the Born Infeld action, that makes explicit calculations more complicated.

As in section 2.4 the zero modes and fluctuations of the BPS states are restricted by the equation

$$\{\bar{Q}, Q\}\, \epsilon = 0, \tag{2.103}$$

where Q are the supercharges of supersymmetric Born-Infeld theory, and ϵ is a 16 component spinor. The restriction on the zero modes can be rewritten with the help of a projection operator [92]

$$(|K| + K_i \tilde{\gamma}_i)\epsilon = 0, \tag{2.104}$$

for some suitable γ matrices. Here K_i is again the five vector (2.54). In the $\alpha' \to 0$ limit this condition reduces to the condition (2.52), which can be seen from the inverse of the 5 dimensional metric (2.98), which collapses to a 3 dimensional metric in the $\alpha' \to 0$ limit.

Unlike the Yang-Mills case the BPS equations of Born-Infeld theory implied by conditions (2.103) and (2.104), are in general quite complicated. However, in the special case that the flux configuration consists of only non-zero rank N and momentum p_3, they lead to the simple chiral 2D free field equations (2.70) [92]. Any configuration of electric and magnetic fluxes can be mapped to this simple case by an appropriate $SL(5, \mathbb{Z})$ transformation (see reference [17] in [92]). This transformation will also act on the gauge fields and the rank N, that can be combined into an antisymmetric matrix of the form (2.53). The component of the 5×5 matrix that is supposed to represent the rank of a gauge group may therefore be non-constant in the transformed matrix. Then the relation to a gauge theory seems to be obscured. This apparent problem can be solved, however, by noting that the BPS equations only depend on the quotients of the gauge fields and rank N. As argued in [92] this observation makes it possible to use $SL(5, \mathbb{Z})$ transformations to generate BPS solutions for general fluxes.

The procedure just sketched extends to the BPS quantum theory [92]. The degeneracy formula for the irreducible states (the long string states) thus obtained is equal to the one in (2.56) and therefore agrees with the M-theory result.

For the irreducible states (the long string states) both theories give the same counting formula (2.56) for the multiplicities. A difference appears when we consider the second quantized BPS spectrum. Single BPS states can be combined into a second quantized BPS state, provided these single states preserve the same supersymmetries. In BI theory the condition (2.103) implies that all 10 charges of the single BPS states align; in SYM only 9 charges have to align. An alternative way to arrive at this conclusion is by considering the BPS energy (mass) spectra. The energies of BPS states add up when they are combined into a new BPS state. For SYM theory this precisely means that 9 charges align (not p_3, when κ_i is turned in the three direction); for BI theory all 10 charges have to align, because all fluxes come in quadratic form in the BPS energies.

We thus see that Born Infeld gives more restrictions on the charges of single BPS states to combine into a second quantized BPS state, than SYM does. As expected the

BI conditions are relaxed and become equal to conditions in SYM, in the $\alpha' \to 0$ limit. We therefore conclude that SYM contains more second quantized states than BI does.

Despite the success in finding appropriate string U-duality symmetries in supersymmetric gauge theories, it is not clear whether these symmetries are actually genuine duality symmetries of the Born Infeld theories. The problem one has in proving electromagnetic duality for $\mathcal{N} = 4$ SYM theory appears here as well: it is difficult to find an explicit duality mapping for the fields. Another problem arises when one investigates Nahm duality, namely the rank N is not the zero mode of any field in the gauge theory. It may be possible to embed the 3+1 BI theory into a six dimensional string theory in the light-cone gauge. One could then think of N as the p^+-momentum of this theory. For a discussion of ideas in this direction see [92].

Chapter 3

High energy scattering in matrix string theory

3.1 Introduction

High energy processes in string theory were first considered from the point of view of conventional string perturbation theory by Gross and Mende [49] in the regime of fixed angle scattering and in the near forward regime by Amati et al in [3]. The recent insights from M-theory, however, have provided a large number of new non-perturbative tools which can now be used to put these works into a new perspective, and extend the results into new directions. For instance, it was long believed that the string length ℓ_s marks the minimal distance that can be probed via scattering processes in string theory. This belief was based on the fact that fundamental strings tend to increase in size when boosted to high energies, and thus appear to be incapable of penetrating substringy distance scales. Since the discovery of D-particles as non-perturbative solitons of the IIA theory, however, we know that there exists small scale structure that, at least for weak string coupling, extends well below the string length [78][23][60][34]. This particular realization provided important motivation for the matrix theory conjecture of [9] that all localized excitations of M-theory (including the fundamental strings) are representable as multi-D-particle bound states [70][87].

In this chapter we begin a study of high energy processes in type IIA string theory, by making use of this matrix theory formalism. We focus on the four graviton scattering amplitude, and in particular we will present a detailed calculation of the pair production rate of D-particles via this process. Our aim is to probe in this way the transition region between the conventional perturbative string regime and the strong coupling regime described by 11-dimensional M-theory (see figure 3.1).

From the ten dimensional perspective of IIA string theory, D-pair production is an inelastic scattering process, in which two strings exchange one unit of D-particle charge. It is inherently nonperturbative and thus inaccessible to conventional perturbative methods. It is also inaccessible in the traditional matrix theory approach since the anti-D particles are boosted to infinite energy.

From the eleven dimensional perspective, on the other hand, the D-pair creation process can simply be thought of as the elastic scattering of two particles in which one unit of Kaluza-Klein momentum in the 11 direction is exchanged. Via this interpretation, one can rather straightforwardly obtain a tree level estimate of the probability amplitude. This estimate should be reliable for large values for the S^1 compactification radius R_{11} and for collision energies sufficiently below the 11-dimensional Planck energy. At high energies and/or small values for R_{11}, on the other hand, we expect the physics of the scattering process to be quite different from (semi-)classical supergravity. In the following we will attempt to gain more insight into this regime via the matrix string approach.

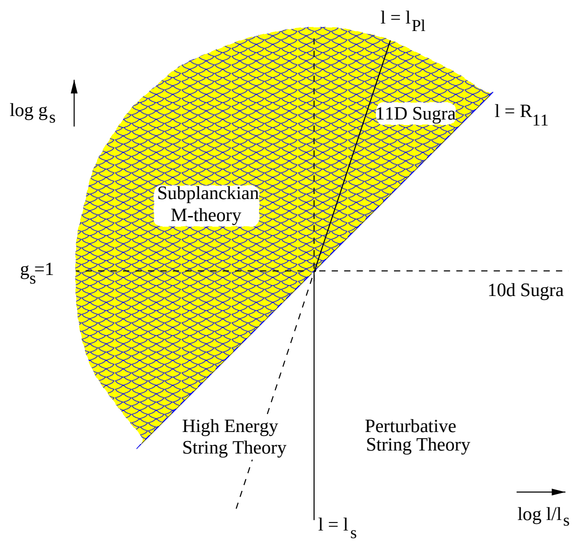

Figure 3.1 The phase diagram of the S^1 compactification of M-theory, with horizontal axis the log of the length scale and vertical axis the log of the string coupling. The various perturbative and low energy limits are indicated. The shaded region marks the regime where D-pair production is expected to be the dominant high energy process.

Matrix string theory arises from the original matrix theory proposal [9] via compactification on a circle, and starts from the action of 1+1-dimensional maximally supersymmetric Yang-Mills theory with gauge group $U(N)$. Via the identification of the eigenvalues of the matrices X^I with the transverse location of type IIA supersymmetric strings, this SYM model can be reinterpreted as a non-perturbative formulation of light-cone gauge IIA string [65][10][29]. In this correspondence, the string coupling constant g_s is inversely proportional to the Yang-Mills coupling g_{YM} (cf. (1.82)) and the free string limit therefore arises in the strong coupling limit of the Yang-Mills model. This correspondence has been worked out in some detail in section 2.2.2.

More generally, however, all regimes of the S^1 compactification of M-theory, as indicated in figure 3.1 should according to the matrix string conjecture of [9][65][10][29] via the above identifications be described by particular regimes of (the large N limit) of the 1+1D supersymmetric gauge theory.

In particular, it is expected that in the weak coupling, moderate energy limit of the SYM theory it effectively reduces to the matrix quantum mechanics description of

11-dimensional supergravity. Indeed, a new feature of matrix string theory (relative to standard light-cone string theory) is that via the electric flux of the gauge field, the string states can be adorned with an extra quantum number, identified with the D-particle charge [29]. In a small g_s expansion, these flux sectors energetically decouple, corresponding to the fact that D-particles can not be produced via perturbative string interactions. Nonetheless, electric flux can get created in the gauge theory: it is easy to see that electric flux creation is a simple one-loop effect that takes place whenever a virtual pair of charged particles gets created and annihilated, after forming a loop that winds one or more times around the σ cylinder.

In the following we will develop a new method for studying high energy scattering and D-pair production in matrix string theory, which will be based on a semi-classical expansion from the SYM perspective. An important novelty of this method is that it applies to processes with arbitrary longitudinal momentum exchange. In the gauge theory language, this means that the transitions between the initial and final states that we will consider will involve a non-perturbative tunneling process in which an arbitrary number of eigenvalues get transferred between the two scattering states. Most previous calculations in matrix theory relied on perturbative SYM corrections and thus were necessarily restricted to zero p^+ transfer. [1]

Concretely, we will construct SYM saddle point configurations that will allow us to interpolate between in-going matrix configurations of the form

$$\vec{X}_{in}(\tau) = \frac{1}{2}\begin{pmatrix} (\frac{\vec{p}_1}{N_1}\tau + \vec{b})I_1 & 0 \\ 0 & (\frac{\vec{p}_2}{N_2}\tau - \vec{b})I_2 \end{pmatrix} \tag{3.1}$$

and outgoing configurations of the form

$$\vec{X}_{out}(\tau) = \begin{pmatrix} (\frac{\vec{p}_3}{N_3}\tau + \vec{b})I_3 & 0 \\ 0 & (\frac{\vec{p}_4}{N_4}\tau - \vec{b})I_4 \end{pmatrix}, \tag{3.2}$$

where I_i are $N_i \times N_i$ identity matrices, where all N_i's are *different* (but subject to the momentum constraint $N_1 + N_2 = N_3 + N_4$). These *in* and *out* configurations each describe two widely separated gravitons with different light-cone momenta

$$p^+_{(i)} = N_{(i)}/R \tag{3.3}$$

and transverse momenta $\vec{p}_{(i)}$, and with relative impact parameter $\vec{b}$.

The interpolating solutions that we will construct, essentially look like an appropriate matrix generalization of perturbative string world-sheets. The importance of these solutions is not entirely obvious, however, since a priori one would expect that the range of validity of the semi-classical Yang-Mills approximation has no overlap

[1] In [73] Polchinski and Pouliot analyzed graviton scattering with non-zero M-momentum transfer in matrix theory. In their case, the M-momentum was identified with the magnetic flux of the SYM gauge theory, and the corresponding instanton was a magnetic monopole. Here we will consider different kind of momentum transfer, namely of longitudinal momentum represented by the size N of the matrix bound states, *i.e.* the number of D-particles in the original matrix dictionary of [9]. This will require a different, less familiar type of instanton process.

with that of perturbative string theory. Indeed, as emphasized in section 2.2.2 the two regimes appear related via a strong/weak coupling duality. However, as we will argue in the following, even at small or moderate string coupling g_s, at sufficiently high collision energies and/or impact parameters one enters a regime in which the semi-classical SYM methods may provide an accurate description of the scattering process.

Just like string/M-theory, the 1+1 SYM model contains various length scales: (i) the circumference of the cylinder, (ii) the scale set by the Yang-Mills coupling $\ell_{_{YM}} = 1/g_{_{YM}}$, in units where the circumference of the cylinder is set equal to 1

$$\ell_{_{YM}} \simeq g_s, \tag{3.4}$$

(iii) the typical mass scale set by the Higgs expectation values of the SYM model. The latter length scale is inversely proportional to the impact parameter b of the string/M-theory scattering process:

$$\ell_b \simeq \frac{g_s}{b}\ , \tag{3.5}$$

in units where $2\pi\alpha' = 1$. Finally, (iv) there is also the length scale ℓ_E determined by the typical size of the SYM energy E, which is related to the relative space-time momenta via $E \simeq p^2/N$.

The existence of these scales allows us to find small dimensionless ratios that may parameterize the strength of the SYM processes taking place at that scale. For example, while $g_{_{YM}} = 1/g_s$ defines the effective coupling of SYM processes that take place at the scale of the YM cylinder, we also have

$$g^{eff}_{_{YM}}(b) \simeq \ell_b/\ell_{_{YM}} \simeq 1/b, \tag{3.6}$$

as the dimensionless coupling at the scale ℓ_b. Similarly, we can also associate an effective coupling $g^{eff}_{_{YM}}(E)$ with the scale set by the SYM energy E. This suggests the possibility that even if g_s is small or of order 1, processes at these other 2D length scales can be accurately described by perturbative and/or semi-classical SYM methods. This will require however that we consider the limit of high collision energies and sufficiently large impact parameters.[2]

The two types of processes that we will consider, high energy scattering with non-zero Δp_+ and the D-pair production, may at first sight seem quite unrelated. However, there are several connections between these two types of processes. First of all, it is worth pointing out that in both cases the scattering process involves (depending on which duality frame one chooses) the transfer of D-particle charge and/or momentum between the two scattering particles. Indeed, the rank N started out as identified with D-particle charge, and only after the duality it and the electric flux E are mapped onto each other under an 11-9 flip: i.e. the interchange of the 11-th and 9-th direction (a

[2]In this context it may be of relevance that in classical 10-dimensional DLCQ supergravity, the impact parameter b scales with the transverse relative momentum p via $b \simeq \left(g_s^2 p^2/N \sin\theta\right)^{1/6}$ with N the DLCQ p_+-momentum. Hence, at least in this classical context, and for fixed scattering angles θ and g_s of order 1, the condition that b is large is automatically satisfied in limit of large $p^2 \gg N$.

detailed explanation on this flip was given in section 2.2.2). Hence quantitative understanding both types of processes will have a direct bearing on the Lorentz invariance of the matrix formalism.

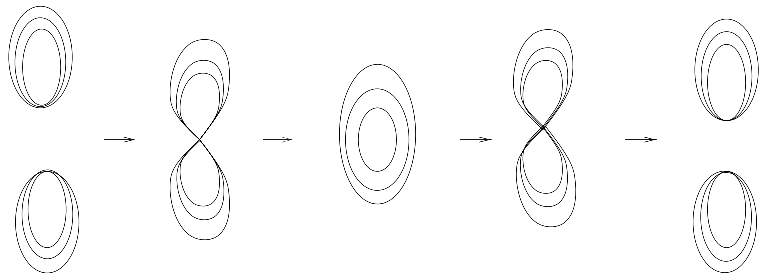

Figure 3.2 This figure depicts a typical saddle point trajectory that contributes to the high energy scattering amplitude of fundamental strings, according to the perturbative physical picture proposed in [49].

Another, more speculative connection between the two calculations is related to a fundamental puzzle in the original calculation of [49], namely the apparently dominant contribution of arbitrarily high genus to scattering amplitudes. The saddle point trajectory at loop order G typically describes a process as depicted in figure 3.2 two incoming strings, that are wound $N = G + 1$ times, interact and then propagate as N intermediate short strings. The N strings then join together again, producing a final state of two different N times wound outgoing strings (see fig. 3.2). It was found in [49] that the contributions of these higher order interactions grows larger with the genus G. This instability appears to signal a fundamental breakdown of conventional string perturbation theory in the high energy regime.

On the other hand, the fragmented form of the intermediate state in figure 3.2 gives a strong hint of some underlying non-perturbative structure that looks quite similar to that of the multi-D-particle bound state dynamics of matrix theory. This suggests that the matrix treatment may provide a rather natural stabilizing mechanism for a cutoff on the genus. Furthermore, our study will show that D-particle pair production becomes relevant at this cutoff – when the strings become maximally fragmented. This leads us to suspect a deeper relation between Gross and Mende's high energy, fixed angle scattering and the non-perturbative process of D-pair creation.

We will begin this chapter with a quick review of the kinematics of fixed angle scattering in the traditional string framework, with particular emphasis on its description in the light-cone gauge. This will be followed by a discussion of the Gross Mende saddle points, first at tree level and then generalized to arbitrary genus g world-sheets. Then we turn to string interactions in the matrix string formulation. In particular, we will find a local instanton solution in the two-dimensional theory that describes the splitting/joining interaction. Furthermore, the condition for this instanton to be matched to the incoming/outgoing states is precisely that we be working at the world-sheet moduli corresponding to the saddle-point surfaces of reference [49].

After this we will briefly discuss recent results of [91], where by a detailed zero mode analysis of the interaction instanton, the four tree-level string scattering amplitude was reproduced exactly, up to a numerical factor.

We then turn to D-pair production, which we consider both from the supergravity perspective as well as via a one loop Yang Mills calculation valid for arbitrary N. This is followed by a discussion of the ranges of validity of our calculations. In particular, we suggest that the ranges of validity of the two calculations overlap, and allow the picture we have mentioned, in which they complement each other. This connection is further discussed in the final section, together with some other observations and speculations.

3.2 Fixed angle scattering of strings

High energy, fixed angle processes in superstring theory were first studied in detail from the point of view of conventional string perturbation theory by Gross and Mende [49]. Central to their approach is the observation that in the limit of large external momenta, the Polyakov path integral at each given perturbative order is dominated by a finite number of saddle point configurations. Furthermore, it was proposed that all these saddle points essentially describe the same preferred world-sheet trajectory, up to an overall factor depending on the loop order.

In the subsequent sections we will find independent evidence from the point of view of matrix string theory that supports this physical picture. In addition we will give a useful characterization of the Gross-Mende saddle points in terms of the light-cone gauge formulation of string perturbation theory.

3.2.1 Kinematic relations for four string scattering

It will be useful to first establish a few kinematic relations of the tree level diagram that describes the scattering of four external massless particles with light-cone momenta $p_i^+ = N_i/R$ and transverse momenta $\vec{p}_i$.

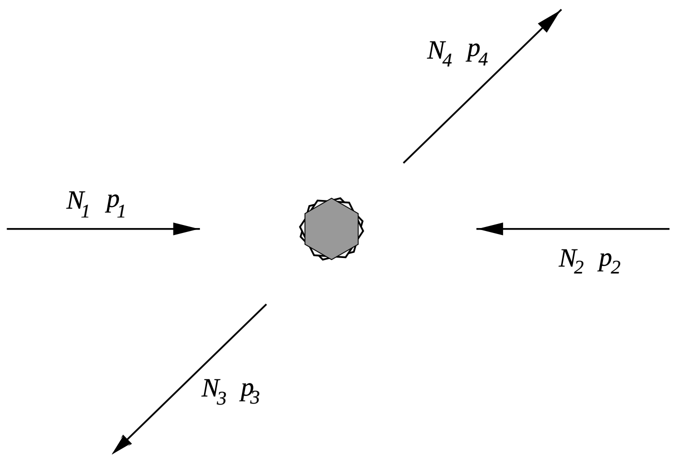

Figure 3.3 This figure indicates the kinematics of the transverse momenta p_i.

For definiteness, we will describe this process in the center of mass frame in the transversal direction

$$\begin{aligned} \vec{p}_1 + \vec{p}_2 &= 0, \\ \vec{p}_3 + \vec{p}_4 &= 0. \end{aligned} \tag{3.7}$$

The four transversal momenta p_i can all be chosen to lie within one given plane. We can thus write all p_i as complex numbers. In addition, longitudinal momentum and energy conservation imply that

$$N_1 + N_2 = N_3 + N_4, \tag{3.8}$$

and

$$\frac{|p_1|^2}{N_1} + \frac{|p_2|^2}{N_2} = \frac{|p_3|^2}{N_3} + \frac{|p_4|^2}{N_4}. \tag{3.9}$$

For a given set of locations z_i of the corresponding vertex operators, the classical location of the world-sheet is described by the equations (1.16) and (1.17)

$$X^+(z,\bar{z}) = \frac{1}{2}\sum_i \epsilon_i N_i \log|z-z_i|^2, \qquad X(z,\bar{z}) = \frac{1}{2}\sum_i \epsilon_i p_i \log|z-z_i|^2. \tag{3.10}$$

In the light-cone gauge, one chooses a fixed world-sheet parameterization by identifying X^+ with the world-sheet time τ, which via (3.10) amounts to setting (after a rescaling)

$$w \equiv \tau + i\sigma = \frac{1}{2\pi}\sum_i \epsilon_i N_i \log(z-z_i) \ . \tag{3.11}$$

The differential $\omega = dw$ (see (1.11)) is a specific globally defined holomorphic differential on the world-surface; existence and uniqueness of such a differential at arbitrary genus [39][38] generalizes the construction to higher loop amplitudes. Notice that (due to the branch cuts in the logarithm) the coordinate σ in (3.11) is defined on an interval $0 \le \sigma < (N_1 + N_2)$, in accord with the rescaling of the world-sheet coordinates with total light-cone momentum p^+.

The light-cone coordinate system (3.11) specifies a particular time-slicing of the string world-sheet. As argued in section 1.1.2, in this coordinate frame there are specific points on the world-sheet at which strings split or join. Namely, these interactions take place at zeros of ω, that is critical points $z = z_0$ of the light-cone coordinate X^+. Inserting the explicit form (1.16) for X^+ then gives the condition (1.19) that can be reduced in the specific case of the four-point scattering amplitude, to an equation relating the interaction point and the Möbius invariant cross ratio

$$\lambda = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}, \tag{3.12}$$

via

$$\frac{N_1}{z_0} + \frac{N_2}{z_0 - 1} = \frac{N_3}{z_0 - \lambda} \ . \tag{3.13}$$

For given λ, this is a quadratic equation for z_0 with in general two solutions z_0^+ and z_0^-, representing the simple splitting and joining interaction respectively.

3.2.2 Gross Mende saddle points

Now we are ready to discuss the Gross-Mende saddle point. Already in chapter one we wrote down the typical (bosonic) g-loop amplitude of n scattering particles in string theory (1.6). The only place where the external momenta p_i enter in the amplitude is the exponential of

$$\mathcal{E}_C(p_i, z_i, m_I) = \frac{1}{2}\sum_{i<j} p_i \cdot p_j G_m(z_i, z_j). \tag{3.14}$$

Here z_i are the insertion points of the vertex operators and G is Green's function associated to the g-loop world-sheet with moduli m. In the limit of large external momenta, the Polyakov path integral is dominated by saddle point configurations that minimize the exponential (3.14). This minimizing problem has an elegant physical interpretation: it is equivalent to finding a semi-stable configuration of two-dimensional Minkowski charges p_i placed at positions z_i on a Riemann surface of genus g and moduli m [49].

In the simplest case of four string scattering at tree level, the Green's function in (3.14) is simply given by the logarithm $G(z_1, z_2) = \log |z_{12}|$ on the plane, so that the Coulomb energy (3.14) becomes

$$\mathcal{E}_C = \frac{1}{2}\sum_{i<j} p_i \cdot p_j \log |z_i - z_j|^2. \tag{3.15}$$

Due to conformal invariance, the energy $\mathcal{E}_C$ depends on the locations z_i of the vertex functions only by means of the cross ratio λ defined by (3.12). The variation of $\mathcal{E}_C$ with respect to this variable λ reads

$$\partial_\lambda \mathcal{E}_C(\lambda) = \frac{p_1 \cdot p_3}{\lambda} + \frac{p_2 \cdot p_3}{\lambda - 1}\ . \tag{3.16}$$

The saddle point equation $\partial_\lambda \mathcal{E}_C = 0$ is solved by

$$\lambda = \frac{p_1 \cdot p_3}{p_1 \cdot p_2} = \frac{t}{s}\ , \tag{3.17}$$

where s, t are the usual Mandelstam variables. The saddle point corresponds to a particular classical world-sheet trajectory which at high energies gives the dominant contribution to the scattering amplitude.

For later reference, it will be useful to translate the above description of the GM saddle point into the light-cone gauge language. To begin with, in the complex parameterization for the p_i, the Mandelstam parameters s and t are expressed as

$$s = -2p_1 \cdot p_2 = (N_1 + N_2)^2 \frac{|p_1|^2}{N_1 N_2}, \tag{3.18}$$

$$t = -2p_1 \cdot p_3 = \frac{|N_3 p_1 - N_1 p_3|^2}{N_1 N_3}, \tag{3.19}$$

so that (3.17) takes the form

$$\lambda = \frac{N_2}{N_3} \frac{|N_3 p_1 - N_1 p_3|^2}{(N_1 + N_2)^2 |p_1|^2}. \tag{3.20}$$

Together with (3.13), this saddle point specifies a particular set of locations for the two interaction points $z_0^\pm$ of the light-cone string diagram. We now claim that this preferred location of the interaction points $z = z_0^\pm$ is singled out by the requirement that, in the immediate neighborhood of $z = z_0^\pm$, the transverse coordinate fields $X(z)$ are (anti-)analytic functions of z

$$\partial_z X \mid_{z=z_0} = 0, \qquad \partial_{\overline{z}} \overline{X} \mid_{z=z_0} = 0. \tag{3.21}$$

To verify this claim, let us compute the cross ratio λ from (3.21). The result should be equal to (3.17). Inserting the solution (1.17) into (3.21) gives

$$\sum_i \frac{\epsilon_i p_i}{z_0^+ - z_i} = 0. \tag{3.22}$$

In terms of the cross-ratio λ defined in (3.12) this reads

$$\frac{p_1}{z_0^+(z_0^+ - 1)} = -\frac{p_3}{z_0^+ - \lambda}, \tag{3.23}$$

where we used that $p_1 + p_2 = 0$. When combined with the equation (3.13), which relates λ with the location of the interaction points z_0, this equation can be used to compute λ in terms of the scattering data. If we subtract N_3 times (3.23) from p_3 times (3.13), we obtain a linear equation for z_0, solved by

$$z_0^+ = \frac{N_1 p_3 - N_3 p_1}{(N_1 + N_2) p_3} \ . \tag{3.24}$$

Further, from (3.23) we find that

$$\lambda = z_0^+ \left(1 + \frac{p_3}{p_1}(z_0^+ - 1)\right) \ . \tag{3.25}$$

After inserting (3.24) into (3.25), it is a simple calculation to verify that the resulting expression for λ coincides with the high energy saddle point (3.17). Note that for the saddle-point configuration, λ is in fact real. The interaction points z_0^+ and z_0^- are in this case each others complex conjugate.

We conclude that for Gross-Mende saddle point world-sheets the transverse coordinate fields behave holomorphically near the interaction points. This light-cone characterization of the GM saddle point in terms of the holomorphicity conditions (3.21) will be critical in establishing the implementation of high energy scattering in the matrix string context.

3.2.3 Higher genus contributions

In general extending the results to higher genus is not straightforward. Especially finding the dominating saddle points will not be so easy. The one-loop case is still relatively simple. Using the explicit form of the Green function on the torus, one can again extremize the electrostatic energy (3.14) of four charges placed on the torus. The resulting extremized energy is a factor two smaller than the tree level energy [49], and surprisingly the saddle point is exactly the same as in the tree level case.

There is little hope to do successful analogous calculations for higher genus diagrams. However in [49], Gross and Mende proposed the following attractive generalization of the saddle-point to higher orders in the string perturbation expansion. They assume that the dominant saddle points at genus g take the form of an $N = g + 1$ fold cover of the same four-punctured sphere as described above, branched over the four locations z_i of the vertex operators. The resulting surfaces are known as $\mathbb{Z}_N$ curves. Such a $\mathbb{Z}_N$ curve is defined by the polynomial expression

$$y^N = \prod_{i=1}^{L} (z - z_i)^{L_i}. \tag{3.26}$$

This curve represents an N sheeted covering of the complex plane, with L branch points of order $N-1$ (provided $\sum_i L_i = 0 \mod N$ and the L_i's relatively prime with respect to N). In case of four string scattering $L = 4$.

When we place electric charges p_i at the branch points the total electric field takes the same value on all N sheets, and is given by

$$E_N(z) = E_N^x - iE_N^y = \frac{1}{N} \sum_i \frac{p_i}{z - z_i}. \tag{3.27}$$

It is straightforward to show that (3.27) is the right expression for the electric field. It is a conservative field and it moreover satisfies Gauss's law. The last property can be easily checked by considering a cycle that encloses a region that contains a branch point. This cycle will encircle the branch point N times, one time on each sheet. Thus the contribution of "surface" integral of the electric field will be equal to the charge enclosed. So the field defined in (3.27) obeys Gauss's law.

The electrostatic energy is given by

$$\mathcal{E}_N = -\frac{1}{2N} \sum_{i<j} p_i \cdot p_j \log |z_i - z_j|^2, \tag{3.28}$$

that is $1/N$ times the energy of the tree level case. Obviously extremizing (3.28) with respect to the Möbius invariant λ will yield the same saddle point (3.17) we found before. This confirms the earlier mentioned claim that at all orders the scattering is dominated by the same saddle point (3.17).

For the $\mathbb{Z}_N$ curves the classical spacetime surface swept out by the strings at loop level N is

$$X^\mu = \frac{1}{2N} \sum_i \epsilon_i p_i^\mu \log |z - z_i|^2, \tag{3.29}$$

where $\epsilon_i = \pm 1$ for incoming respectively outgoing strings. We see that the coordinate fields grow with increasing energy, so that strings cannot be used as probes to explore string theory at small distances (high momenta). The classical trajectory of a higher order saddle point (3.29) has the same shape as the tree level trajectory, but its size is N times smaller. The intuitive reason is that they describe multiple wound strings, so that the effective string tension is N times bigger than usual. Correspondingly, since the different trajectories are weighted by the world-sheet area, the higher order trajectories give contributions proportional to $e^{-\mathcal{E}_C/N}$ (with $\mathcal{E}_C$ defined by (3.15)). The higher genus contributions are thus quite strongly enhanced at high energy.

It is worth pointing out that the structure of the $\mathbb{Z}_N$ curves and the corresponding space-time trajectories, as depicted in figure 3.2, are quite reminiscent of the description of the "long string" boundary conditions in section 1.2. In our view, this (proposed) structure of the higher order interactions is one of several indications that the Gross-Mende approach to high energy string scattering may have a natural implementation in the matrix string context.

3.3 Matrix string interactions

In this section we will prepare the ingredients for the semi-classical study of high energy scattering in the matrix string framework. To begin with, we notice that the above light-cone gauge description of the dominant string world-sheet trajectories can rather easily be put into a matrix form, by the procedure introduced in section 1.1.4. Starting from equations (1.16) and (1.17), we represent the classical string trajectory by means of a diagonal $N \times N$ matrix (with $N = N_1 + N_2$) by first writing the transversal coordinates $\vec{X}$ as a function of w defined in (3.11), and then "roll up" the spatial interval $0 \le \sigma < N$ onto the short interval $0 \le \sigma < 1$. Concretely, we define the diagonal matrix elements of $X(\sigma)$ via $X_{kk}(\sigma) = X(\sigma + k)$, and in this way we indeed create matrix configurations that, away from the interaction times, satisfy the long string boundary condition (1.29) and (1.30).

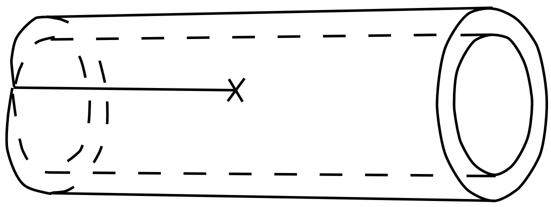

Figure 3.4 The string interaction relating a one string to a two string state. This interaction occurs when two eigenvalues X_I and X_J coincide, we enter a phase where an unbroken $U(2)$ symmetry is restored. Figure taken from [89].

These diagonal matrix configurations represent particular solutions to the SYM equations of motion, that are regular everywhere *except* at the interaction points. If at some point in the (σ, τ) plane two eigenvalues X_I and X_J coincide, we enter a phase where locally the gauge symmetry is restored to $U(2)$. In general we should thus expect that in this local region the semi-classical SYM solution will need to become truly non-abelian.

It is readily seen that the diagonal matrix configurations constructed via the above procedure from the CFT solution (1.17) is not single valued around the interaction points. Instead, as explained in section 1.3.1, in going around the interaction point, the matrix X undergoes a simple transposition of the two degenerating eigenvalues. This transposition changes the long string boundary conditions and therefore corresponds to an elementary string interaction.

In the gauge theory language, the diagonal CFT solution (1.17) in fact hides a delta-function Yang-Mills curvature at the interaction point, such that the infinitesimal Wilson line around it coincides with this permutation group element [89]. In this section we will describe how the Yang-Mills dynamics smoothes out this singularity.

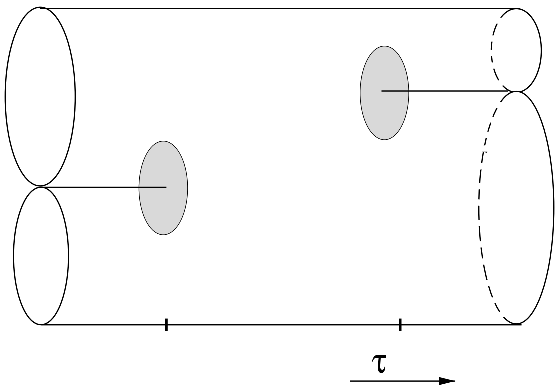

Figure 3.5 The shaded areas are the interaction regions where the singularity of the abelian string configuration is smeared out by an instanton-like solution. The diagram corresponds to a four string scattering process at tree level.

Concretely, we will now exhibit a smooth and single-valued Yang-Mills configuration that describes the local splitting or joining of one or two matrix strings. The idea is to glue this solution into the abelian CFT configuration in a neighborhood of the interaction points (see figure 3.5). The conditions we have to impose for this gluing procedure will help us to find the Yang-Mills configuration.

Ultimately, we will be interested in obtaining global classical solutions to the SYM equations of motion that minimize the Yang-Mills action for given asymptotic conditions on the matrix fields X, as written in equations (3.1) and (3.2) in the introduction.

3.3.1 SYM Solution near interaction point

It seems reasonable to assume that, at least in the immediate neighborhood of the interaction point, these minimal action configurations of the SYM model are described by supersymmetric configurations. Hence, instead of trying to solve the full Yang-Mills equations, we will restrict ourselves to the special class of solutions satisfying a dimensionally reduced version of the self-duality equations from four to two dimensions. We will choose to work with complex variables

$$X = \frac{1}{2}(X^1 + iX^2)\ ,\ \overline{X} = \frac{1}{2}(X^1 - iX^2)\ , \tag{3.30}$$

setting the remaining X^i's to zero. The self-duality conditions then become

$$F_{w\overline{w}} = -\frac{i}{g_s^2}[X, \overline{X}],$$
$$D_w X = 0, \quad D_{\overline{w}}\overline{X} = 0.$$

The above equations are most conveniently analyzed by writing

$$\begin{aligned} A_w(w,\overline{w}) &= -iG\partial_w G^{-1}, \\ A_{\overline{w}}(w,\overline{w}) &= i(\partial_{\overline{w}}\bar{G}^{-1})\bar{G}, \end{aligned} \tag{3.31}$$

where $G(w,\overline{w})$ denotes an element of the *complexified* ($\bar{G} \neq G^{-1}$) gauge group. This parameterization of A_α allows one to solve the second and third equation of (3.30), via

$$X(w,\overline{w}) = G\widehat{X}(\overline{w})G^{-1} \ . \tag{3.32}$$

The first equation in (3.30) then takes the following form

$$\partial_{\overline{w}}(\Omega\partial_w\Omega^{-1}) = -\frac{1}{g_s^2}[\Omega\widehat{X}(\overline{w})\Omega^{-1}, \widehat{\overline{X}}(w)] \tag{3.33}$$

with

$$\Omega = \bar{G}G. \tag{3.34}$$

Let us now look at the local neighborhood of an interaction point. For convenience, we choose coordinates such that it is located at $w = 0$. Since the interaction involves only two eigenvalues, it is sufficient to consider only the corresponding $SU(2)$ part of the matrices. The matrix $\widehat{X}$, which parameterizes the local coordinate distance between the two interacting strings, can be chosen of the following form

$$\widehat{X}(\overline{w}) \simeq \pm B\,\sqrt{\overline{w}}\,\tau_3, \tag{3.35}$$

for some constant B. The $\pm$ indicates that the interaction point $w = 0$ represents a square root branch point for the diagonal matrix $\widehat{X}$ in (3.35), which therefore is multi-valued.

The diagonal matrix $\widehat{X}(\overline{w})$, together with $A = 0$, represents a valid solution of the SYM equations (3.30) except at the interaction point, where analyticity fails. Therefore we will look for a true solution of the form (3.32), where $G \to 1$ asymptotically far from $w = 0$. A helpful *Ansatz* for $G(w,\overline{w})$ is

$$G = e^{\frac{1}{2}\alpha\,\tau_1}, \tag{3.36}$$

where for $\alpha(w,\overline{w})$ we choose a real function (so that $G = \bar{G}$ and $\Omega = \exp(\alpha\tau_1)$) that tends to zero far away from the interaction point. We now compute

$$\Omega\widehat{X}\Omega^{-1} = B\sqrt{\overline{w}}e^{\alpha\tau_1}\tau_3 e^{-\alpha\tau_1} = B\sqrt{\overline{w}}\begin{pmatrix}\cosh 2\alpha & -\sinh 2\alpha \\ \sinh 2\alpha & -\cosh 2\alpha\end{pmatrix}. \tag{3.37}$$

Hence

$$\left[\Omega\widehat{X}\Omega^{-1},\widehat{\overline{X}}\right] = 2|B|^2|w|\ \sinh 2\alpha\ \tau_1 \tag{3.38}$$

and thus we find that under the present Ansatz the equation of motion (3.33) reduces to

$$\partial_w\partial_{\overline{w}}\alpha = \frac{2}{g_s^2}|B|^2|w|\sinh 2\alpha, \tag{3.39}$$

which is essentially the familiar sinh-Gordon equation. (It can be transformed to the exact sinh-Gordon equation after a (multi-valued) coordinate transformation $w \to \tilde{w} = w^{3/2}$.)

The boundary condition that we must impose on $\alpha(w,\overline{w})$ at $w=0$ follows from the requirement that the Yang-Mills configuration be regular. This condition is most easily understood in the gauge where X is *single-valued* near $w=0$; in this gauge the YM curvature $F_{w\overline{w}}$ should be a regular function at $w=0$. The configuration (3.32)–(3.36), however, is (for single-valued and real α) multi-valued. We can make X single-valued by applying the singular gauge transformation

$$\begin{array}{ll} X & \to UXU^{-1}, \\ A_w & \to -iUD_wU^{-1}, \end{array} \tag{3.40}$$

with gauge parameter

$$U = e^{\pm i\theta\tau_1/4}, \tag{3.41}$$

with $\theta = \frac{1}{2i}\log(w/\overline{w})$ the azimuthal angle around $w=0$. In this gauge

$$\begin{array}{rcl} A_w & = & i\Big[\frac{1}{2}\partial_w\alpha \pm \frac{1}{8w}\Big]\ \tau_1, \\ A_{\overline{w}} & = & -i\Big[\frac{1}{2}\partial_{\overline{w}}\alpha \pm \frac{1}{8\overline{w}}\Big]\ \tau_1. \end{array} \tag{3.42}$$

Using that $\partial_w\frac{1}{\overline{w}} = \pi\delta^{(2)}(w)$, this gives

$$F_{w\overline{w}} = -i\tau_1\Big(\partial_w\partial_{\overline{w}}\alpha \pm \frac{\pi}{4}\delta^{(2)}(w)\Big). \tag{3.43}$$

The regularity requirement at $w=0$ is therefore that $\partial_w\partial_{\overline{w}}\alpha \simeq \mp\frac{\pi}{4}\delta^{(2)}(w)$. We thus deduce that the solution to equation (3.39) that we want must satisfy the following asymptotic condition

$$\alpha(w,\overline{w})\ \simeq\ \mp\frac{1}{2}\log|w| + const. \qquad w\to 0, \tag{3.44}$$

while at large distances from the interaction point α must tend to zero.

Now let us write $\alpha=\alpha(r)$ with $r=|w|$. The equation of motion (3.39) reduces to the ordinary non-linear differential equation

$$(\partial_r^2 + \frac{1}{r}\partial_r)\alpha = \frac{8}{g_s^2}|B|^2 r\sinh 2\alpha\ . \tag{3.45}$$

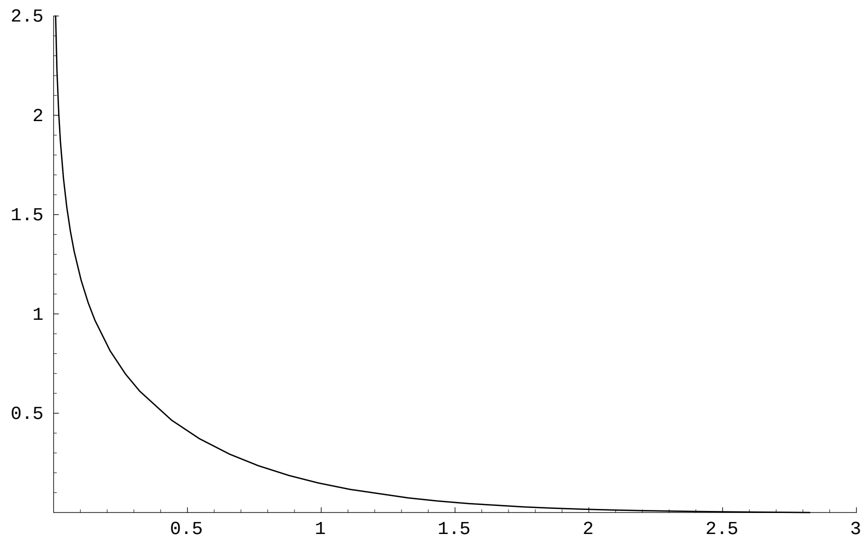

Figure 3.6 The numerical solution $\alpha(s)$ to $(\partial_s^2 + \frac{1}{s}\partial_s)\alpha = s \sinh 2\alpha$, as a function of the distance s from the interaction point. The initial condition for small s, that after integrating leads to the correct asymptotic behavior for large s, reads $\alpha(s) = -\frac{1}{2}\log s + 0.0305070(1)$.

A numerical solution to this equation is depicted in figure 3.6. [3] For large $r = |w|$ the solution looks like

$$\alpha(r) \sim \frac{\mp 1}{|w|^{3/4}} \exp\Big(-\frac{8|B|}{3g_s}|w|^{3/2}\Big) \ . \tag{3.46}$$

From this asymptotic behavior we read off the typical instanton size

$$\ell_{\rm inst} \sim g_s^{2/3}|B|^{-2/3}. \tag{3.47}$$

3.4 High energy scattering of matrix strings

The matrix solution of the string interaction constructed in the previous subsection, should be viewed as a local description in the immediate neighborhood of the interaction point. In general, it must therefore be glued via an appropriate patching procedure into a complementary CFT type solution (*e.g.* as described in section 3.2) that matches with the asymptotic scattering data at the far past and future. The idea here is that (as we will see shortly) at sufficiently high collision energies, the size of the interaction regions are small compared to the rest of the matrix string world-sheet. Hence, while the behavior (3.35) provides the asymptotics for large $|w|$ at the UV scale of the matrix solution, it also provides the local boundary condition near the interaction point for the CFT solution for X that describes the IR part of the saddle point.

[3] We thank Pierre van Baal for pointing out an error in the original figure in [40].

The solution (3.30) that we described is not the most general SYM description of a string splitting and joining event, but rather the most symmetric one, with smallest action. This means therefore that there is a non-trivial matching condition on the corresponding matrix string world-sheet: from (3.35) we see that we must require that the transverse string coordinates X behave holomorphically near the interaction point. Remarkably, this is exactly the same condition as (3.21), which tells us that the shape of the string world-sheet must be precisely that of the Gross-Mende saddle point! Therefore these solutions seem appropriate to a YM generalization of the high-energy scattering of [49]. In this section, we fill in a few more details of this connection.

3.4.1 Evaluation of the classical action

In order to estimate scattering amplitudes via the instanton processes, one must calculate the instanton action. The bosonic part of the SYM action (with only two X-fields non-vanishing) can be written as

$$S = \int d^2w \left\{ -g_s^2 \Big(F_{w\overline{w}} + \frac{i}{g_s^2} [X, \overline{X}] \Big)^2 + 4 D_w X D_{\overline{w}} \overline{X} \right\} + \oint (\overline{XD}X + XD\overline{X} - \overline{X}DX - X\overline{DX}) \tag{3.48}$$

and thus for the supersymmetric configuration that satisfy (3.30), the total classical action reduces to a boundary term

$$S_{cl} = \oint (\overline{X\partial}X + X\partial\overline{X}) \tag{3.49}$$

identical to the boundary term needed to glue the non-abelian matrix solution described in the previous section into the CFT type solution. Hence we claim that, in the limit that the matrix interaction points become sufficiently small, the SYM action for the above saddle point configurations coincides with the CFT action, *i.e.* for the case of a tree level string diagram it equals the "Coulomb energy" (3.15), where me must insert the saddle-point value for locations z_i. It is perhaps worth pointing out that this saddle point actions is fully Lorentz invariant, as it should be. While this is not surprising once we have established the connection with the GM saddle point, it does seem to represent a rather non-trivial statement from the SYM point of view!

More generally we see that from (3.48) we can derive (as usual) an inequality, which suggests that whenever the interaction does not take place at a holomorphic point for the X-fields, the SYM action is always larger than the corresponding CFT action. This provides additional evidence for the conjecture that the above type of configurations represent dominant saddle-points, that minimize the SYM action.

Obviously, there exist a large number of CFT-type solutions for which X varies (anti-) holomorphically near all interaction points. In particular, there are the higher genus $\mathbb{Z}_N$-curves of [49]. In addition it is also possible to write down SYM solutions that describe multiple string world-sheets, but nonetheless still satisfy the appropriate boundary conditions, as specified in equations (3.1) and (3.2) in the introduction. Ideally, one would like to know which (sub-class) of these solutions provide the truly dominant contribution to the scattering amplitude.

3.4.2 Minimal distance

The parameter $|B|$ that governs the size of the interaction vertex, as seen in (3.46), can be straightforwardly determined in terms of the momenta of the external states. The coordinate system $(w, \overline{w})$ on the Yang-Mills cylinder that we used in the analysis of the self-dual Yang-Mills equation (3.30), coincides with the light-cone coordinates defined in (3.11). From this and (3.35) we immediately find

$$|B|^2 \;=\; 2\frac{|\partial_{\overline{z}} X(z_0^+)|^2}{|\partial_z^2 X^+(z_0^+)|} \,. \tag{3.50}$$

A straightforward calculation then gives

$$|B|^2 = \frac{|p_1\overline{p}_3 - \overline{p}_1 p_3|(N_1+N_2)}{\sqrt{N_1 N_2 N_3 N_4}} \,. \tag{3.51}$$

It is interesting to note that for this solution, even though the eigenvalues of the complex coordinate matrix X vanish at the interaction point, the full matrix coordinate X in fact does not! Instead, near $w = 0$ it approaches the constant non-diagonalizable matrix

$$X(w, \overline{w}) \;\simeq\; const.\; g_s^{1/3} B^{2/3} \begin{pmatrix} 1 & \mp 1 \\ \pm 1 & -1 \end{pmatrix} \qquad w \to 0. \tag{3.52}$$

The value of the overall constant can be determined numerically. From this we read off that the minimal "distance" between the two interacting strings is in fact non-zero! Instead, we have

$$d_{min} = \sqrt{\mathrm{tr}(X(0)\overline{X}(0))} \sim g_s^{1/3}|B|^{2/3}. \tag{3.53}$$

Although it is tempting to speculate (as indeed we will do in the concluding section), the precise physical significance of this result is as yet unclear to us. We do notice, however, that the typical world-sheet size ℓ_{inst} of the matrix interaction region, as can be read off from (3.46), is naturally expressed in terms of this minimal relative distance as $\ell_{inst} = (g_s/|B|)^{2/3} = g_s/d_{min}$.

3.5 One loop fluctuation analysis

In principle it is now possible to do a computation of the one-loop determinant of the quantum fluctuations around the semi-classical saddle-point configurations. An important motivation for performing such an analysis is to obtain a semi-classical estimate for the absolute strength of the splitting and joining interactions in matrix string theory. Duality symmetries of M-theory give the precise prediction that this strength should be governed by the string coupling g_s. To verify this, one needs to compare the SYM one-loop determinant with the Gross-Mende fluctuation determinant, coming from the Gaussian integration over the Riemann surface moduli around the saddle-point.

This comparison has been done for tree level fixed angle scattering in the high energy limit [91]. The result is that all of the expected structure of the high energy scattering amplitude is reproduced, including the power of the string coupling constant g_s and the kinematical factor.

When one does an expansion around a classical background one has to do two things: a calculation of the fluctuation determinant and a separate treatment of the zero modes. In our case the fluctuation determinant is approximately equal to one, because the background field is taken to be self-dual in a neighborhood of the interaction points. Then the fermionic and bosonic contributions cancel each other. Note however that non self-dual corrections to the background will change this simple result.

In the approximation where the fluctuation term is equal to one, the zero modes determine the form of the scattering amplitude. Below we will discuss them briefly. For an extensive and detailed treatment we refer to the work of Wynter in [91].

3.5.1 Zero modes of the instanton

The strategy of the zero mode analysis of [91] is based on the philosophy we sketched in section 3.3 : We have no explicit solution of the full YM saddle-points, but we know their form near the interaction points and their asymptotics far away from the interaction points. The instanton type solutions near the interaction points satisfy boundary conditions that are compatible with the abelian matrix string configurations. These configurations consist of commuting matrix fields, together with an abelian gauge field that generates the appropriate monodromies around the interaction points [89]. Their singular behavior at the interaction points is smoothened out by the instanton solutions. The same is true for the non-trivial zero modes: the abelian matrices have non-trivial zero modes with singularities at the interaction points. These singularities are, however, smoothened out by the instanton zero modes.

To make this more explicit, we will now review the zero modes of the instanton solutions. For this, it is convenient to write them in the multi-valued gauge

$$\begin{aligned} A_w &= \frac{1}{2} i\partial_w \alpha\, \tau_1, \\ X &= B\sqrt{\bar{w}}\,[\cosh\alpha\, \tau_3 - i\sinh\alpha\, \tau_2]\,. \end{aligned} \tag{3.54}$$

We distinguish between four types of bosonic zero modes [91], namely those associated to translations and deformations of the instanton solution (the zero modes of the fields X and $\bar{X}$), gauge field zero modes, the zero modes of the six transversal coordinates X^I and those of the ghost fields.

All bosonic zero modes satisfy the eigenvalue equation for quadratic fluctuations around the classical instanton background which reads (after inclusion of the background gauge fixing term $(\bar{D}_\mu A_\mu)^2$ to the action)

$$D^2 V_\mu + F_{\mu\nu} V_\nu = 0. \tag{3.55}$$

Here we are using a ten dimensional notation; D^2 and $F_{\mu\nu}$ are taken with respect to the instanton background field.

The gauge field zero modes originate from the left-over gauge degree of freedom; they can be written as pure gauges, $V_w = D_w \Lambda$. They will prove to be of particular interest in the following, because they determine the g_s coupling constant dependence of the contribution to the scattering amplitude at g-loop order [18][91]. Plugging the expression for the instanton and the pure gauge V_w in equation (3.55), we get the following condition for Λ

$$D_w D_{\bar{w}} \Lambda + D_{\bar{w}} D_w \Lambda - \frac{1}{g^2}([X,[\bar{X},\Lambda]] - [\bar{X},[X,\Lambda]]) = 0. \tag{3.56}$$

Solutions to this equation are Λ_n and $(\Lambda_n)^*$ [91], given by

$$\Lambda_n = \begin{cases} Bw^n(\cosh\alpha\ \tau_3 - i\sinh\alpha\ \tau_2) & n = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots \\ w^n \mathbb{1} & n = 1, 2, 3\dots \end{cases} \tag{3.57}$$

Of particular relevance is the zero mode for which Λ coincides with the X-field of the instanton solution

$$V_{\bar{w}} = \frac{1}{2} B\bar{w}^{-1/2}(\cosh\alpha\ \tau_3 - i\sinh\alpha\ \tau_2). \tag{3.58}$$

Though this zero mode has a branch starting at the origin, it is single-valued when we go back to the gauge (3.41). Away from the interaction region it behaves as an *abelian* zero mode with $\bar{w}^{-1/2}$ asymptotic behavior

$$V_{\bar{w}} \to \frac{1}{2} B\bar{w}^{-1/2}\tau_3 \qquad (\bar{w} \to \infty). \tag{3.59}$$

Hence the zero mode (3.58) can be glued into the zero mode of an abelian gauge field that is defined on the light-cone cylinder, except for an interaction point, where it has a singularity. One can remove this singularity by going to the double cover. As explained in section 1.1.2 the light-cone cylinder can be lifted to a double cover in a neighborhood of an interaction point via $w = z^2$. On the Riemann surface the abelian zero mode becomes

$$V = \bar{w}^{-1/2} d\bar{w} = d\sqrt{\bar{w}} = d\bar{z}. \tag{3.60}$$

Together with the other solutions (3.57) we get a complete set of abelian gauge field zero modes in the complex z-plane

$$V_{\bar{z}} = 1, \bar{z}, \bar{z}^2, \bar{z}^3, \dots \tag{3.61}$$

The zero modes for the bosonic fields X^1 and X^2 correspond to translations of the interaction points (branch points of the light-cone cylinder). An obvious guess for the translation mode is

$$V_\mu = \partial_\alpha A_\mu - D_\mu A_\alpha = F_{\alpha\mu}, \tag{3.62}$$

which can be easily verified to be a solution of equation (3.55) (α runs over the light-cone coordinates). The first term in (3.62) corresponds to a translation in the α-direction, the second term is a gauge transformation which forces the zero mode to satisfy the background gauge condition.

Its asymptotic behavior is

$$V_w \to \bar{w}^{-1/2}\tau_3, \tag{3.63}$$

which is again smoothened out at the origin of the instanton solution in the regular, single-valued gauge.

The remaining bosonic zero modes are those of the Higgs fields X^I (where now $I = 3 \ldots 8$), which correspond to translations in the (six) transverse directions; and the ghost zero modes. Both types of zero modes satisfy the equation (3.56) [91], so the functions (3.57) form a basis for them. However there is no solution with asymptotic $\bar{w}^{-1/2}$ behavior that is finite at the origin (in the regular gauge). Hence we have no non-trivial zero modes for the ghost fields and the Higgs fields.

Finally we have fermionic zero modes. They are associated to the unbroken supersymmetry of the instanton configuration. There is one zero mode with abelian asymptotic $\bar{w}^{-1/2}$ behavior, which is finite at the origin [91].

Now we know the local behavior of the zero modes in a neighborhood of the interaction points. They are smooth at the interaction points, and behave like abelian modes away from these points. This last property makes it possible to glue them into the zero modes of the matrix string configurations. These were constructed in [91]. We will not discuss them here; instead we refer to reference [91].

We conclude that not only the field configurations of the instantons can be glued into the abelian background field, but their non-trivial zero modes as well.

3.5.2 Tree level high energy scattering

For the four string scattering process illustrated in figure 3.5 the calculation of the tree level amplitude amounts to a careful treatment of the bosonic and fermionic zero modes.

Integration of the bosonic zero modes corresponding to simple translations of the string configuration in the six transverse directions X^I, gives rise to transverse momentum conservation. The non-trivial zero modes which are associated to simultaneous translations of the two branch points, in the σ respectively τ direction, lead to invariance under shifts in σ respectively conservation of light-cone energy p^-. Relative displacements of the two branch points in figure 3.5 are somewhat more involved. A careful integration over these zero modes leads to a kinematic factor in the scattering amplitude. This contribution has the form [91]

$$\frac{c}{sut} e^{-\frac{1}{4}(s \log s + t \log t + u \log u)}, \tag{3.64}$$

where c is a constant. The expression (3.64) is a familiar term from the calculation by Gross and Mende of tree level high energy scattering [49].

The integration over the ghost zero modes and the gauge field zero modes together yield a factor g_s^2, the correct string coupling constant dependence for a four string scattering amplitude at tree level.

Finally the fermionic zero modes give rise to the right kinematical factor. Thus the high energy limit of the string theory scattering amplitude is reproduced by matrix string theory, up to a numerical factor. This is an encouraging result; it gives clear evidence for the matrix string theory conjecture of section 2.2.2. As a further check one should compare loop contributions with the results of Gross and Mende as well. In principle this calculation can be performed, but the large N limit should be taken with great care (see also [90]).

We end this section with by remarking that, as was originally argued in [18], the g_s coupling dependence is reproduced by matrix string theory for any light-cone Riemann surface. The power of the string coupling is determined by integration over the gauge field zero modes and the ghost field zero modes. Each (abelian) gauge field zero mode contributes a positive power of g_s. The constant ghost zero mode yields a factor g_s^{-1} [18]. This reasoning is justified by the analysis of Wynter, who verified the local existence of the zero modes around the interaction points.

Than it is just a matter of counting the number of independent abelian gauge zero modes or equivalently the number of independent cycles on the Riemann surface. When we start with a tree level diagram with n external states, we have $n-1$ independent cycles. For each loop (handle) we glue to the diagram, we have to add two cycles, so that we have $2g+n-1$ abelian gauge field zero modes for a g genus light-cone diagram with n external states. There is only one constant ghost field zero mode, thus we come to the conclusion that the string coupling dependence is $g_s^{2g+n-2} = g_s^{-\chi}$ where χ is the Euler number of the Riemann surface. This is the right power of g_s, known from string theory [49].

This concludes our discussion of the one-loop fluctuation analysis.

It seems even more worthwhile to look for true new physical effects that might arise from the one-loop corrections. Compared to the conventional perturbative string description, the new degrees of freedom in matrix string theory are the charged components of the X-fields, as well as the extra gauge potential A_α. These new degrees of freedom are non-perturbative from the string perspective, and their quantum fluctuations could thus potentially lead to new physics. As we will show in the next section, there is indeed such a new effect: the pair creation of D-particles.

3.6 D-Particle pair production

In this section we turn to the process of pair creation of D charge, which is in our description x^9 momentum or equivalently (under the matrix string duality) electric flux. This can be viewed as a contribution to the fluctuations about the high-energy scattering processes of the preceding sections, or as a process worthy of interest in its own right in the context of graviton scattering. There are several viewpoints from which this can be investigated. In the limit where x^9 decompactifies, this simply matches onto the standard supergravity calculation [15]. In fact, we can work backwards from this, using the method of images, and compute the amplitude at large finite R_9, in the special situation with source and probe particles, $N_1 \gg N_2$. We will discuss this calculation first. Alternately, one can study this process directly in the matrix string approach, and derive the pair-production rate via a one-loop Yang Mills calculation. This latter approach gives a leading order result valid for arbitrary N_1 and N_2, and also more readily makes connection with the other results of this chapter. Furthermore, the Yang-Mills calculation also apparently extends beyond the region where supergravity is a valid approximation.

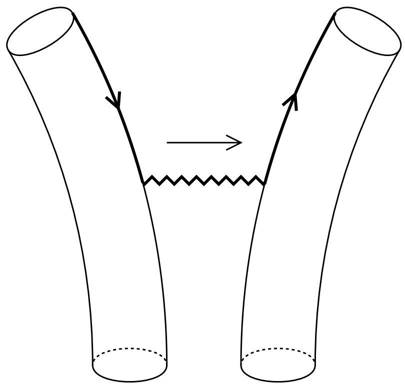

Figure 3.7 In matrix string theory strings can be adorned with an extra quantum number, the D-particle number. This enables one to compute non-perturbative corrections to perturbative string scattering amplitudes. The figure illustrates the creation/annihilation of a D-particle pair, a process that is reminiscent of electron-positron creation/annihilation.

3.6.1 Supergravity calculation

Consider 11-dimensional supergravity compactified on $S^1 \times S^1$, where one S^1 a light-like circle of radius R

$$x^- \equiv x^- + 4\pi R \tag{3.65}$$

and the other S^1 denotes a space-like circle of radius R_9. As we have seen in section 2.2.2, in the M-theory/matrix string correspondence this second radius R_9 is expressed as $R_9 = g_s$ in string units (cf. (2.21)).

Consider in this set-up the scattering process of two massless particles of light-cone momenta $p_i^+ = N_i/R$ and transverse momenta p_i. Let us first consider the probe situation $N_2 \ll N_1$. Then one can already get quite useful information about the scattering process from considering the classical gravitational force between the two particles. The boosted particle with $p_+ = N_1/R$ produces via its stress-energy a non-trivial gravitational background, described by the generalized Aichelburg-Sexl shock-wave geometry of the form [2][14]

$$ds^2 = -dx^-(dx^+ + f(r,a)dx^-) + dx_\perp^2 + g_s^2 da^2, \tag{3.66}$$

with

$$f(r,a) = \sum_k \frac{-15N_1 g_s^3}{2R^2(r^2 + g_s^2(2\pi k + a)^2)^{7/2}}. \tag{3.67}$$

Here a denotes the coordinate distance from the gravitational source in the compact x^9 direction, and the sum over k arises from the image points in this direction.

The momentum four vector of the second massless particle moving in this background geometry will satisfy a dispersion relation of the form

$$2p^-(p^+ + f(r,a)p^-) = p^2 + e^2/g_s^2, \tag{3.68}$$

where e denotes the quantized momentum in the x_9 direction. We can solve for the light-cone Hamiltonian p^- of the particle and obtain

$$p^- = \frac{p^+}{2f(r,a)}\left\{\sqrt{\left(1+\frac{2f(r,a)}{(p^+)^2}(p^2+e^2/g_s^2)\right)}-1\right\}. \tag{3.69}$$

Substituting $p^+ = N_2/R$, (and rescaling the light-cone time by a factor of R) we can write this as

$$H = H_0 + H_{int}, \tag{3.70}$$

where

$$H_0 = \frac{1}{2N_2}(p^2 + e^2/g_s^2), \tag{3.71}$$

and

$$H_{int} \simeq -\frac{15N_1 g_s^3}{8N_2^3}\sum_k \frac{(p^2+e^2/g_s^2)^2}{\left(r^2+g_s^2(2\pi k+a)^2\right)^{7/2}} + \ldots\,. \tag{3.72}$$

Hence the motion of the second particle in terms of the light-cone time x^+ looks like that of a particle with mass N_2 moving in $R^8 \times S^1$ under the influence of an interaction potential given by (3.72).

From this description we can now quite easily extract a low energy prediction for the D-pair production rate. To this end, it is useful to rewrite the interaction Hamiltonian via a Poisson resummation as

$$H_{int} \simeq -\frac{1}{2\pi}\frac{N_1}{N_2^3}g_s^2(p^2+e^2/g_s^2)^2\sum_n \exp(ina)\int dT T^2\exp(-Tr^2)\exp(-n^2/4g_s^2T). \tag{3.73}$$

The $n = 1$ term in this series is the term that corresponds to changing the compact momentum by one unit, *i.e.* to D-charge production. Working to first order in perturbation theory, we can then compute the corresponding phase shift, using

$$\delta = -\int d\tau H_{int}(b^2 + p^2\tau^2)\ , \tag{3.74}$$

where b is the impact parameter and τ the light-cone time.

3.6.2 D-PAIR PRODUCTION VIA ELECTRIC FLUX CREATION

We now study this problem of D-charge creation in the matrix string framework. More generally, we consider scattering states which asymptotically have momenta of the form

$$p^\mu = (p^-, \vec{p}, p_9 = n/R_9, p^+ = N/R) \ . \tag{3.75}$$

These include both gravitons ($n = 0$) and D0-branes – or anti-branes – ($n = \pm 1$). The case of current interest begins with an initial state of two gravitons, and pair produces a D particle pair. This process is intrinsically non-perturbative from the point of view of string theory. It is also a process not accessible in the standard matrix theory approach, where the anti-branes are boosted away to infinite energy.

In principle (for example on a sufficiently large computer) it appears possible to calculate such amplitudes to arbitrary order in the coupling $g = g_{YM} = 1/g_s$, and calculate the D-pair production rate even for small g_s. In the coming sections we will work to leading non-trivial order (one-loop), and leave further calculations to other work. Similar calculations have been performed in the context of matrix theory in [13].

3.6.3 D-PARTICLES IN MATRIX STRING THEORY

As we have indicated in chapter 2, an important new feature of the matrix string theory formalism (relative to standard light-cone string theory) is that via the electric flux, string states can be adorned with a non-vanishing D-particle charge. In this subsection we will describe this in somewhat more detail.

To add to this interpretation, let us first show that each separate string can carry only one type of electric flux. Consider a single long string in matrix string theory with length N. Define the $U(N)$ matrix U such that

$$UV = VUe^{\frac{2\pi i}{N}}, \tag{3.76}$$

with V the usual cyclic permutation matrix on the N eigenvalues defined in (1.30). Hence we can take

$$U = \begin{pmatrix} 1 & & & \emptyset \\ & e^{\frac{2\pi i}{N}} & & \\ & & \ddots & \\ \emptyset & & & e^{\frac{2(N-1)\pi i}{N}} \end{pmatrix} . \tag{3.77}$$

The $SU(N)$ part of the electric flux in this sector is defined as

$$\hat{U} |\, \psi_e \,\rangle = \exp\Big(\frac{2\pi i e}{N}\Big) |\, \psi_e \,\rangle, \tag{3.78}$$

with $e \in \mathbf{Z}_N$ and $\hat{U}$ the quantum operator that implements the constant gauge rotation

$$(A, X) \to (UAU^{-1}, UXU^{-1}). \tag{3.79}$$

Since diagonal matrices are inert under this gauge rotation, we conclude that the $SU(N)$ part of the electric flux dynamically decouples from the diagonal matrix string configurations (1.23) that describe the separate freely propagating strings. Now recall that in $U(N)$ SYM theory, the overall $U(1)$ part of the electric flux is related to the $SU(N)$ part e via

$$\mathrm{tr}E = e \quad (\mathrm{mod}\ N)\ . \tag{3.80}$$

Supersymmetry ensures that the ground state in the $SU(N)$ sector has zero energy even for $e \neq 0$. Hence the total ground state energy receives only a contribution from the overall $U(1)$ flux. In the following we will thus identify e with the total $U(1)$ electric flux. From the above description it is further clear that we can turn on only one electric flux per long string, as is appropriate for its identification with D-particle charge.

The energy of the ground state in this electric flux sector is equal to

$$H_0 = \frac{e^2}{2Ng_s^2}. \tag{3.81}$$

General ground state configurations

$$|N^{(i)}, p_\perp^{(i)}, e^{(i)}\rangle \tag{3.82}$$

of s separate strings of individual length $N^{(i)}$, transverse momenta $p_\perp^{(i)}$, and D-particle charge $e^{(i)}$ have a SYM energy equal to

$$H_0 = \sum_{i=1}^{s} \frac{1}{2N^{(i)}}\Big[(p_\perp^{(i)})^2 + (e^{(i)}/g_s)^2\Big], \tag{3.83}$$

which, when rescaled by R, is the sum of the p_- light-cone momenta of the corresponding collection of string ground states

$$\sum_{i=1}^{s} \frac{1}{2p_+^{(i)}}\Big[(p_\perp^{(i)})^2 + (M^{(i)})^2\Big]. \tag{3.84}$$

In particular, we read off from (3.83) that the states with D-particle charge $e^{(i)}$ each have mass

$$M^{(i)} = \frac{e^{(i)}}{g_s} = \frac{e^{(i)}}{R_9}, \tag{3.85}$$

in accordance with their identification as graviton states with non-zero KK momentum in the compact direction.

3.6.4 ONE-LOOP CALCULATION, $N = 2$

For simplicity we begin with the case where the incoming and outgoing particles all have $N = 1$. The next subsection will generalize to arbitrary N. The asymptotic states take the form

$$\bar{X}^1 = \frac{1}{2}\begin{pmatrix} p\tau & 0 \\ 0 & -p\tau \end{pmatrix}\ ,\ \bar{X}^2 = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\ , \tag{3.86}$$

corresponding to two particles with center of mass momentum p and impact parameter b (measured in string units). It will also be useful work with a non-trivial gauge background

$$\bar{A}_\sigma = \begin{pmatrix} a/2 + e\tau/g_s^2 & 0 \\ 0 & -a/2 - e\tau/g_s^2 \end{pmatrix},\ \bar{E} = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}\ . \tag{3.87}$$

The constant electric field corresponds to a non-zero D-charge for the incoming and outgoing particles, with quantization

$$e \in \mathbb{Z}\ . \tag{3.88}$$

The prototypical example of production of D-charge is in processes where this changes by one unit,

$$\Delta E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{3.89}$$

Introducing the constant background potential a will help keep track of such changes.

In the large string coupling/small Yang-Mills coupling limit, the leading contribution to D-charge producing processes is easily computed via a one-loop super-Yang Mills calculation. Calculations at higher loop order then give subleading corrections in $g = 1/g_s$.

Our starting point is the Yang-Mills action (1.81), although it will be simpler to begin with it written in its un-dimensionally reduced form in terms of the gauge field

$$A^M = (A^\mu, gX^i), \tag{3.90}$$

with $M = 0, \cdots, 9$, $\mu = 0, 9$ and $i = 1, \cdots, 8$. We will decompose the gauge field into background and fluctuation pieces,

$$A_M = \bar{A}_M + g\tilde{A}_M\ . \tag{3.91}$$

The Feynman background gauge-fixed Lagrangian is

$$\mathcal{L} = -\mathrm{Tr}\left\{\frac{1}{4g^2}(F_{MN}^2) + \frac{1}{2g^2}(\bar{D}^M A_M)^2 - i\bar{\psi}\bar{D\!\!\!/}\psi\right\}, \tag{3.92}$$

where $\bar{D}_M = \partial_M + i\bar{A}_M$. Using the decomposition (3.91) we find

$$\mathcal{L} = -\mathrm{Tr}\left\{\frac{\bar{F}^2}{4g^2} + \frac{1}{2}(\bar{D}_M\tilde{A}_N)^2 + i\bar{F}^{MN}[\tilde{A}_M, \tilde{A}_N] - i\bar{\psi}\bar{D\!\!\!/}\psi + \right. \tag{3.93}$$

$$g\bar{\psi}\tilde{A}\!\!\!/\psi + ig\bar{D}_M\tilde{A}_N[\tilde{A}^M,\tilde{A}^N] - \frac{g^2}{4}[\tilde{A}_M,\tilde{A}_N]^2\Bigg\}.$$

The amplitude in question is given by

$$\mathcal{A}(a,e) = \int \mathcal{D}\tilde{A}_\mu \mathcal{D}\tilde{X}^i \mathcal{D}\psi e^{iS}, \tag{3.94}$$

where the boundary conditions on the functional integral are chosen to correspond to the asymptotic behavior given in (3.86),(3.87).

If we write

$$A_M = \frac{1}{2}A_{Ma}\sigma^a = \frac{1}{2}A_{M+}\sigma^+ + \frac{1}{2}A_{M-}\sigma^- + \frac{1}{2}A_{M3}\sigma^3, \tag{3.95}$$

with $\sigma^\pm$ defined in terms of the usual Pauli matrices via

$$\sigma^\pm = \frac{\sigma^1 \pm i\sigma^2}{\sqrt{2}}, \tag{3.96}$$

then the couplings in (3.93) include the standard charged minimal couplings of A_+, A_-, ψ_+, and ψ_- to the U(1) field $\tilde{A}_{\mu 3}$. The amplitude to create unit electric flux is therefore given by summing the loop contributions to (3.94) in which one of these charged particles circulates once around the σ-direction; higher encirclings yield more flux. Therefore we need the contribution of the charged state windings to the one-loop amplitude.

This immediately follows by reading off the spectrum from the second through fourth terms of (3.93) in the backgrounds (3.86) and (3.87). We begin by defining

$$\hat{p}^2 = g^2p^2 + 4g^4e^2\ . \tag{3.97}$$

In the bosonic sector we find the massless, neutral fields

$$\begin{array}{lll} \tilde{X}^3_i, & i=1,\cdots,8; & m^2=0 \\ A^3_\mu, & & m^2=0 \end{array} \tag{3.98}$$

and the charged fields

$$\begin{array}{rcc} \tilde{X}^\pm_i\ , i=2,\cdots,8 & ; & m^2 = r^2 \equiv \hat{p}^2\tau^2 + a^2 + g^2b^2 \\ \frac{1}{\hat{p}}(gp\tilde{A}^{9\pm} - 2eg^2\tilde{X}^{1\pm}) & ; & m^2 = r^2 \\ \tilde{A}^{0\pm} + \frac{i}{\hat{p}}(2eg^2\tilde{A}^{9\pm} + gp\tilde{X}^{1\pm}) & ; & m^2 = r^2 + 2\hat{p} \\ \tilde{A}^{0\pm} - \frac{i}{\hat{p}}(2eg^2\tilde{A}^{9\pm} + gp\tilde{X}^{1\pm}) & ; & m^2 = r^2 - 2\hat{p}. \end{array} \tag{3.99}$$

For the fermions, we have 16 massless uncharged states, 8 charged states with masses $m^2 = r^2 + \hat{p}$, and 8 charged states with masses $m^2 = r^2 - \hat{p}$, as in [13]. Finally, including the ghosts gives one complex, uncharged field C^3 with $m^2 = 0$ and one complex charged field $C^{\pm}$ with $m^2 = r^2$.

All of the charged fields are minimally coupled to the background field $\bar{A}_9 \equiv \bar{A}_\sigma$. At one loop level, we have

$$\mathcal{A}_1(\mathrm{a},\mathrm{e}) = \int \prod_{\mathrm{I}} \mathcal{D}\Phi_{\mathrm{I}} \mathrm{e}^{\mathrm{i}\mathrm{S}^{(2)}[\Phi_{\mathrm{I}}]}, \tag{3.100}$$

where I labels the charged fields enumerated above (the uncharged contributions cancel), arbitrary winding is allowed, and where $S^{(2)}$ is the quadratic part of the action (3.93) including the coupling to $\bar{A}^3_\sigma$ through $\bar{D}$. Working with phase shifts, we then have

$$\begin{aligned} i\delta_1 = \ln\mathcal{A}_1 = \sum_I \ln \sum_n \int_n \mathcal{D}\Phi_I e^{iS^{(2)}[\Phi_I]} = \\ -\sum_I (-1)^{F_I} \sum_n \int_0^\infty \frac{dS}{S} \int_n \mathcal{D}\tau\mathcal{D}\sigma \, \exp i \int_0^S ds \big[\dot{\sigma}^{\mu 2}/2 - \bar{A}^3_\mu \dot{\sigma}^\mu(s) - m_I^2(\tau)/2\big]. \end{aligned} \tag{3.101}$$

Here we have used the functional integral representation in terms of the first-quantized trajectory $\sigma^\mu(s) = (\tau(s), \sigma(s))$, n is the winding number about the cylinder, and F_I denotes fermion number of the field.

For general winding n the functional integrals in (3.101) are readily rewritten in terms of functional determinants. For example, with $m^2 = r^2$ we have

$$\int_n \mathcal{D}\tau\mathcal{D}\sigma \, e^{i\int_0^S ds\left[\dot{\sigma}^{\mu 2}/2 - g^2(p^2\tau^2+b^2)/2 - \bar{A}^3_\sigma\dot{\sigma}(s)\right]} = e^{-ina + in^2/2S - ig^2b^2S/2}\Delta(p,e,S), \tag{3.102}$$

where

$$\Delta(p,e,S) = \det{}^{-1/2}\begin{pmatrix} \partial_s^2 - g^2p^2 & -2g^2e\partial_s \\ 2g^2e\partial_s & -\partial_s^2 \end{pmatrix} . \tag{3.103}$$

Combining such expressions and defining $S = 2T$ then gives

$$i\delta_1(a,p,e) = \sum_n e^{-ina} \int_0^\infty \frac{dT}{T} e^{in^2/4T - ig^2b^2T} \Delta(p,e,T)\Big[-6 - 2\cos(2gpT) + 8\cos(gpT)\Big]. \tag{3.104}$$

Recalling the quantization rule (3.88), we see that as long as $gp \gg 1$ the electric background only contributes at higher order in g; neglecting this, the determinant is readily evaluated using

$$\det{}^{\frac{1}{2}} i(-\partial_s^2 + g^2p^2) \;\; = \;\; gp\prod_{n=1}^{\infty}\left[\left(\frac{2n\pi}{S}\right)^2 + g^2p^2\right] = 2\sinh\left(gpT\right), \tag{3.105}$$

$$\det{}^{\frac{1}{2}} i(-\partial_s^2) \;=\; \sqrt{\frac{2\pi T}{i}}\ .$$

Here we used ζ-function regularization.

The phase shift then becomes

$$i\delta_1(a,p,e) = \frac{1}{2}\sqrt{\frac{i}{2\pi}}\sum_n e^{-ina}\int_0^\infty \frac{dT}{T} e^{in^2/4T - ig^2b^2T}\frac{1}{\sqrt{T}\sinh(gpT)} \times\Big[-6 - 2\cos(2gpT) + 8\cos(gpT)\Big]. \tag{3.106}$$

In the full amplitude $\mathcal{A}_1 = \exp\{i\delta_1\}$, the coefficient of the term e^{-ina} is the amplitude to make a transition from a state with electric flux e to $e+n$:

$$\mathcal{A}_1(a,e) = \sum_n e^{-ina}\langle e+n|e\rangle\ . \tag{3.107}$$

We have found that this is independent of e to order g^2. The amplitude for a change by one unit of charge (e.g. two gravitons to $D\bar{D}$ pair), as well as the effective interaction Hamiltonian, can be derived from these expressions in the range $p \ll b^2$. There the integrand in (3.106) can be expanded in pT to find

$$i\delta_1(a,p,e) \approx -\frac{g^3p^3}{2}\sqrt{\frac{i}{\pi}}\sum_n e^{-ina}\int dT T^{3/2} e^{in^2/4T - ig^2b^2T}\ ; \tag{3.108}$$

the leading order $D\bar{D}$ production amplitude is just the coefficient of e^{-ia} in the series. From (3.108) and (3.74) we can also work backwards to extract the effective Hamiltonian. We find

$$H_{int} = -\frac{15}{8}g^{-3}p^4\sum_k \frac{1}{[r^2 + (a+2\pi k)^2/g^2]^{7/2}}\ , \tag{3.109}$$

in agreement with (3.72).

3.6.5 Generalization to arbitrary N

The one-loop calculation of the preceding subsection is readily generalized to the case where the incoming and outgoing particles have arbitrary (though discretized) p_{11}, or equivalently, N. In this case there are a variety of different boundary conditions that may be placed on the $N\times N$ blocks. Two that we will consider are the trivial boundary condition,

$$X(\sigma+1) = X(\sigma)\ , \tag{3.110}$$

and the single long string boundary condition,

$$X(\sigma+1) = V^{-1}X(\sigma)V, \tag{3.111}$$

where V is given in (1.30).

In the case of two incoming states with momenta N_1, N_2, we write

$$X^i = \bar{X}^i + \tilde{X}^i, \tag{3.112}$$

where X^i is an $(N_1 + N_2)\times(N_1 + N_2)$ matrix. In particular, the background is taken to be

$$\bar{X}^1 = \frac{p\tau}{2}\begin{pmatrix} I/N_1 & 0 \\ 0 & -I/N_2 \end{pmatrix} \equiv \frac{p\tau}{2}T_D \qquad \bar{X}^2 = \frac{b}{2}T_D, \tag{3.113}$$

where we have split the matrix into $N_1\times N_1$ and $N_2 \times N_2$ blocks corresponding to two "clusters," and I represents the corresponding identity matrices.

A useful decomposition of the fluctuations $\tilde{X}^i$ is in terms of the matrices

$$T^{a_1} = \begin{pmatrix} t^{a_1} & 0 \\ 0 & 0 \end{pmatrix}, \; T^{a_2} = \begin{pmatrix} 0 & 0 \\ 0 & t^{a_2} \end{pmatrix}, \tag{3.114}$$

where t^{a_i} are hermitian generators of $SU(N_i)$; $T_+^{\alpha_1\alpha_2}, T_-^{\alpha_1\alpha_2}$, which have matrix elements

$$\begin{aligned} (T_1^{\alpha_1\alpha_2})_{\beta_1\beta_2} &= \sqrt{2}\delta_{\alpha_1\beta_1}\delta_{N_1+\alpha_2\beta_2}, \\ (T_2^{\alpha_1\alpha_2})_{\beta_1\beta_2} &= \sqrt{2}\delta_{N_1+\alpha_1\beta_1}\delta_{\alpha_2\beta_2}; \end{aligned} \tag{3.115}$$

and T_D:

$$\tilde{X}^i = \frac{\tilde{X}_D}{2}T_D + \frac{\tilde{X}_{a_1}}{2}T^{a_1} + \frac{\tilde{X}_{a_2}}{2}T^{a_2} + \frac{\tilde{X}_{+\alpha_1\alpha_2}}{2}T_+^{\alpha_1\alpha_2} + \frac{\tilde{X}_{-\alpha_1\alpha_2}}{2}T_-^{\alpha_1\alpha_2}. \tag{3.116}$$

Following the preceding subsection (and working with $e = 0$ for simplicity) we find that the charged states now have an extra N_1N_2 in their multiplicities, and have masses

$$\begin{aligned} \tilde{X}^i_{\pm\alpha_1\alpha_2} : \quad m^2 &= \tfrac{r^2}{4\nu^2} \\ \tilde{A}^9_{\pm\alpha_1\alpha_2} : \quad m^2 &= \tfrac{r^2}{4\nu^2} \\ \tilde{A}^0_{\pm\alpha_1\alpha_2} + \tilde{X}^1_{\pm\alpha_1\alpha_2} : \quad m^2 &= \tfrac{r^2}{4\nu^2} + \tfrac{gp}{\nu} \\ \tilde{A}^0_{\pm\alpha_1\alpha_2} - \tilde{X}^1_{\pm\alpha_1\alpha_2} : \quad m^2 &= \tfrac{r^2}{4\nu^2} - \tfrac{gp}{\nu} \end{aligned} \tag{3.117}$$

where

$$\frac{1}{\nu} = \frac{1}{N_1} + \frac{1}{N_2}. \tag{3.118}$$

Likewise, the charged fermions and ghosts have masses as in the $N = 2$ case with the trivial rescalings to

$$\bar{p} = \frac{p}{2\nu}, \; \bar{b} = \frac{b}{2\nu}. \tag{3.119}$$

Therefore, in the case of trivial boundary conditions the amplitude (and Hamiltonian) is exactly as computed in (3.106) with the only difference being multiplication by N_1N_2 and replacement of p and b as in (3.119). Note that $2\bar{p} = \frac{p}{N_1} + \frac{p}{N_2}$ is simply relative velocity of the two clusters, and $\bar{b} = \frac{1}{2}(\frac{b}{N_1} + \frac{b}{N_2})$ is precisely the impact parameter between the clusters.

In the case of long-string boundary conditions, this result is modified. Now

$$X(\sigma + 2\pi) = \begin{pmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{pmatrix} X(\sigma) \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \tag{3.120}$$

and in particular the charged off-diagonal blocks satisfy twisted boundary conditions

$$\begin{aligned} X_+(\sigma+1) &= V_1^{-1} X_+(\sigma) V_2, \\ X_-(\sigma+1) &= V_2^{-1} X_-(\sigma) V_1 \ . \end{aligned} \tag{3.121}$$

The matrices V can be diagonalized by working on basis vectors

$$w_k = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ \lambda^k \\ \lambda^{2k} \\ \vdots \\ \lambda^{(N-1)k} \end{pmatrix} \ , \ \lambda = e^{2\pi i/N}, \ k \in \mathbb{Z} \tag{3.122}$$

and in this basis simply give phases λ^k. Thus the amplitude (3.106) is modified to

$$\begin{aligned} i\delta_1 = \frac{1}{2}\sqrt{\frac{i}{2\pi}} \sum_{\alpha_i=1}^{N_i} \sum_n \ & e^{-in[2\pi(\alpha_1/N_1 - \alpha_2/N_2)+a]} \int_0^\infty \frac{dT}{T} e^{in^2/4T - ig^2\bar{b}^2 T} \frac{1}{\sqrt{T}\sin(g\bar{p}T)} \\ & \times \Big[-6 - 2\cos(2g\bar{p}T) + 8\cos(g\bar{p}T) \Big] , \end{aligned} \tag{3.123}$$

and the interaction Hamiltonian takes the form

$$H_{int} \approx \frac{g^4 p^4}{2\pi i} \sum_{n,\alpha_1,\alpha_2} e^{-in[2\pi(\alpha_1/N_1 - \alpha_2/N_2)+a]} \int dT T^2 e^{in^2/4T - ir^2 T} \ . \tag{3.124}$$

For non-zero n, the supergravity correspondence no longer holds when $N > 2$: the matrix string then yields a different result. In (3.124) the expression in the summation only gives a non-zero contribution when the integer n is a multiple of both N_1 and N_2. Hence we see that the minimal exchanged D-particle number between two long strings of length N_1 and N_2 must be proportional to N_1N_2 (if the lengths are relatively prime), else the amplitude will be simply zero.

This leads us to the conclusion that the long strings do not give an effective means of creating D-particles. For two strings to create a minimally charge D-pair, the strings apparently first need to each emit a minimal length string such that two short strings of both collections can exchange a single D-particle. In the SYM language, this last process is effectively an $SU(2)$ process, where correspondence with 11-dimensional supergravity is found. It is important to note, however, that the electric flux thus created must then subsequently spread out over the complete $U(N_i)$ gauge group, since otherwise it would not carry the SYM energy appropriate for the massive D-particle with $p^+ = N_i/R$.

Furthermore, in the sector with a fixed p^+ momentum, we now have an improved idea of what state contributes most to D-charge production: it is the state with trivial boundary conditions, (3.110), corresponding to a collection of minimal length strings. Since this is the state that yields amplitudes agreeing with low-energy supergravity in the limit $g \to 0$, it is apparently this state (or a bound version of it when finite g effects are taken into account) that dominates the wave-function of the graviton in the small g region, rather than the state with the long string boundary conditions (3.111).

3.6.6 Ranges of validity

In this section we will give a preliminary discussion of the relevant scales and ranges of validity of the calculations of the preceding sections. This analysis is preliminary in that the systematics of the perturbation theory for the Yang-Mills Lagrangian (1.81) has not been performed at the level of that for pure matrix theory [14] and additional subtleties are possible. We leave such analysis for future work. For simplicity we will consider the case where the p^+ momentum of the two incoming states are comparable, $N_1 \sim N_2 \sim N$. Our arguments readily generalize to the probe situation $N_1 \gg N_2$.

We begin by considering the expansion of the action about a classical background as in (3.93); such a treatment is relevant both for corrections to the saddle-point solutions of section four as well as for the systematic treatment of D-pair production.

This expansion is governed by the Yang-Mills coupling g_{YM}, and naively one expects the condition $g_{YM} \ll 1$ for corrections to be small. However, as mentioned in the introduction, the Yang-Mills coupling is scale dependent and one expects the relevant scale to be set by the physics one is considering.

For example, in the scattering with background (3.86), loops of the charged, massive states of the YM theory play a central role. One either has a loop localized on the cylinder, whose calculation leads to the $\mathcal{O}(v^4/r^6)$ supergravity potential, or the loop can encircle the cylinder leading to the D-pair production that we have computed. These massive states receive masses of minimum size b/g_s through the Higgs effect, setting the length scale $\ell_b \simeq g_s/b$. At this scale, we expect the relevant dimensionless parameter to be

$$g_{YM}\ell_b \simeq \frac{1}{b} \; . \tag{3.125}$$

Smallness of this parameter thus requires

$$b \gg 1 \; . \tag{3.126}$$

For the case of pair creation, there is another requirement arising from the condition that the back reaction due to the created electric field be small. One way of stating this is to require that the YM energy be large as compared to the energy stored in the electric field,

$$p^2 \gg g_{YM}^2 \ . \tag{3.127}$$

Finally, in the case of the string interactions of section four, we see from (3.46) that the relevant scale is set by the parameter $|B|$, and is given by

$$\ell_{inst} \simeq \left(\frac{g_s}{|B|}\right)^{2/3} \simeq \left(\frac{g_s^2 N}{p^2 \sin\theta}\right)^{1/3} \ . \tag{3.128}$$

At this scale the dimensionless coupling is given by

$$g_{YM}\ell_{inst} \simeq \left(\frac{N}{g_s p^2 \sin\theta}\right)^{1/3} \ . \tag{3.129}$$

Another condition to apply the methods of section four is that the size of the instanton be small as compared to the size of the cylinder, $\ell_{inst} \ll 1$, or

$$p^2 \gg \frac{g_s^2 N}{\sin\theta} \ . \tag{3.130}$$

It is certainly possible to simultaneously satisfy the conditions (3.126), (3.127) and (3.130), as well as the more stringent condition $g_{YM} \ll 1$, for finite N and large $s \sim p^2$. If all important corrections are governed by expansions in the parameters of (3.125) and (3.129), then it appears possible to even push the calculations into the range $g_s \lesssim 1$.

A more complete analysis can be performed in the large g_s (large R_9) case in the restricted energy range

$$\frac{1}{g_s} \ll E \ll g_s \ . \tag{3.131}$$

The lower bound corresponds to the energy threshold to create D charge, and the upper bound is the energy to create winding states wrapping x^9. In between these bounds the theory can be effectively described by matrix theory DLCQ quantized in 10 dimensions.

As explained in [14], the matrix expansion is an expansion in terms of the form

$$\left(\frac{N}{M_{pl}^3 r^3}\right)^L \left(\frac{v^2}{R^2 M_{pl}^6 r^4}\right)^n , \tag{3.132}$$

where L counts loops and M_{pl} is the eleven dimensional Planck mass. The terms with $L = n$ are readily identified with terms in the corresponding supergravity expansion, and the small parameter justifying this expansion is [14][8]

$$\frac{Nv^2}{R^2 M_{pl}^9 r^7} \ll 1. \tag{3.133}$$

This has a simple physical interpretation, which is easily seen by estimating the net transverse momentum transfer due to the potential

$$\frac{N^2 v^4}{R^3 r^7} \; ; \tag{3.134}$$

this gives

$$\Delta p_\perp = \frac{N^2 v^3}{R^3 b^7} \; . \tag{3.135}$$

The condition (3.133) is then easily seen to be $\Delta p_\perp \ll p$, or equivalently $\theta \ll 1$ where θ is the scattering angle.

Expansion terms with $n > L$ are then suppressed for

$$p^2 \ll g_s^{-2} N^2 r^4 \; , \tag{3.136}$$

and terms with $n < L$ for

$$r \gg (N g_s)^{1/3} \; . \tag{3.137}$$

It is unclear whether the latter condition is strictly necessary; the first term in this expansion vanishes [13][14], and the other terms have been conjectured to vanish in [8].

To better understand these conditions, we convert them into statements relating the Mandelstam parameter $s \sim p^2$ and the angle θ. It is easily seen that condition (3.136), using (3.135), becomes

$$s \ll N^2 \left(\frac{N}{\theta}\right)^{4/3} g_s^{-14/3} \tag{3.138}$$

and the condition (3.137) becomes

$$s \gg g_s^{7/3} N^{10/3} \theta \; . \tag{3.139}$$

Comparing (3.130) with (3.138) and (3.139) , we see that within the energy range given by (3.131) the instanton and D-particle production calculations are not obviously simultaneously valid. However by relaxing the restrictions by giving up (3.137) (and (3.139)), which maybe justified due to a non-renormalization theorem, it seems that it is possible that the calculations are simultaneously valid. Outside the energy range (3.131) we appeal to the preceding (less rigorous) analysis which suggests that these calculations are indeed simultaneously valid at large s, and may even be extendable to $g_s \lesssim 1$. It is partly on this basis that we will, in the next section, consider the consequences of combining these two calculations.

3.7 Discussion and conclusions

We begin this section by recalling several observations from our preceding discussion. The first is that, as pointed out in section 3.6.5, string scattering only efficiently produces D charge if the strings break off at least one minimal length string. Furthermore, sections four and five discussed saddle-point configurations that are expected to make important contributions to high energy, fixed angle scattering. Combining these yields a picture of how the important non-perturbative process of D-charge production can arise in high-energy string scattering.

The analysis of Gross and Mende [49] found saddle-points believed to dominate scattering at high energy. These saddle-points have a common structure at arbitrary genus, and the contributions of these saddle-points grows with the genus suggesting the relevance of non-perturbative effects. We have found a new version of their analysis in which a mechanism appears that can cut off this growth. The cutoff originates from the minimal string length, which is in our language the minimal p^+. String fragmentation is stopped when the string breaks into the maximal number of minimal-length strings.

It is precisely in the context of minimal length string scattering that we have found that D-charge pair production can become an important effect. We therefore have a very nice picture in which the instantons of section four and five lead to maximal fragmentation of the strings, and this is followed by the production of D-charge via the process of section six. Here we expect that the size (3.46) of the instanton, as well as the corresponding minimal distance (3.53), may be an important ingredient in determining the size of both these effects.

From the stringy viewpoint this is an intrinsically non-perturbative process. This is suggestive that there is in fact a basic connection between these two processes, and in particular that the non-perturbative production of D-charge is an important correction to the high-energy scattering analysis of [49]. While we believe that, by combining the various ingredients presented in this paper, it may be possible to obtain definite quantitative estimates of these corrections, we leave further analysis of this connection for future work.

Next we turn to several other observations and connections.

First, recall that Banks and Susskind [12] previously considered the $D\bar{D}$ system in the context of perturbative string theory. There they found an instability with unknown outcome. In the present framework we have been able to treat the same system analytically, at least in the large g_s limit, without signs of pathology. In principle, the matrix string calculations appear to extend to arbitrary g_s. One might hope that some extrapolation of our approach could shed further light on the discussion of [12].

It is an interesting conceptual question under which circumstances one needs to include the virtual effects of D-particles propagating in loops. Although in the literal sense of an expansion about $g_s = 0$ they do not contribute, since they have infinite mass there, there is clearly a strong sense in which D-particles can be found in intermediate states when g_s is finite. Indeed, intermediate states with D-charge are distinguished from other intermediate states only by the presence of electric flux, and there is no apparent reason why these should be suppressed at finite g_s. In fact, looking at the results of section 3.6, leads one to suspect that it may be possible to extend the matrix string interactions as discussed in chapter two, to include the possibility of electric flux "pair" creation. The eleven dimensional symmetry of M-theory, in particular, suggests as a possible generalization of the DVV string-interaction vertex, an expression of the form $V_{int} = V_{twist}\delta(A_{12})$ (with A_{12} the difference between the U(1) gauge fields on the two strings that are created). With this choice of vertex, the couplings between string and all n-D-particle bound states are all of the same strength. This would suggest that there may possibly exist a systematic semi-classical expansion in string theory – generalizing the standard perturbative expansion – in which the D-particle-loop contributions play the role of instanton-like corrections. Indeed, in other recent studies of non-perturbative contributions to string scattering amplitudes [44][45][46][48] it was suggested that D-particle loops are related via T-duality to D-instanton contributions in IIB string theory. It clearly would be interesting to see if the suggestive formulas obtained in these works can possibly be reproduced via the matrix string methods developed in this paper.

To conclude, we have succeeded in using the matrix string approach to begin an investigation of aspects of high energy string scattering, and in particular to begin to explore the role of important non-perturbative (from the string viewpoint) processes such as D-charge production. Further investigation along these lines is expected to unravel a rich structure at substringy scales, and may shed further light on the short distance structure and fundamental degrees of freedom and dynamics of M-theory.

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